1.2 Limita funkce

1.2.59 $\lim\limits_{x\to 0}\dfrac{\cos x-\cos 3x}{x^2}$

Řešení:

$$\lim_{x\to 0}\frac{\cos x-\cos 3x}{x^2}=\lim_{x\to 0}\frac{(\cos x-1)+(1-\cos 3x)}{x^2}=\lim_{x\to 0}\frac{\cos x-1}{x^2}+\lim_{x\to 0}\frac{1-\cos 3x}{x^2}=$$
$$=-\lim_{x\to 0}\frac{1-\cos x}{x^2}+\lim_{x\to 0}9\frac{1-\cos 3x}{(3x)^2}=-\frac{1}{2}+9\lim_{x\to 0}\frac{1-\cos 3x}{(3x)^2}=-\frac{1}{2}+9\cdot\frac{1}{2}=\underline{4}.$$

1.2.60 $\lim\limits_{x\to 0}\dfrac{1+\sin x-\cos x}{1+\sin px-\cos px},\quad p\neq 0$

Řešení:

$$\lim_{x\to 0}\frac{1+\sin x-\cos x}{1+\sin px-\cos px}=\lim_{x\to 0}\frac{(1-\cos x)+\sin x}{(1-\cos px)+\sin px}=\lim_{x\to 0}\frac{\dfrac{1-\cos x}{x}+\dfrac{\sin x}{x}}{\dfrac{1-\cos px}{x}+\dfrac{\sin px}{x}}=$$
$$=\lim_{x\to 0}\frac{x\cdot\dfrac{1-\cos x}{x^2}+\dfrac{\sin x}{x}}{p^2x\dfrac{1-\cos px}{(px)^2}+p\dfrac{\sin px}{px}}=\lim_{x\to 0}\frac{\left(x\cdot\dfrac{1-\cos x}{x^2}+\dfrac{\sin x}{x}\right)}{\left(p^2x\dfrac{1-\cos px}{(px)^2}+p\dfrac{\sin px}{px}\right)}=$$
$$=\frac{\lim\limits_{x\to 0}\left(x\cdot\dfrac{1-\cos x}{x^2}\right)+\lim\limits_{x\to 0}\dfrac{\sin x}{x}}{\lim\limits_{x\to 0}\left(p^2x\dfrac{1-\cos px}{(px)^2}\right)+\lim\limits_{x\to 0}\left(p\dfrac{\sin px}{px}\right)}=\frac{\lim\limits_{x\to 0}x\cdot\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}+\lim\limits_{x\to 0}\dfrac{\sin x}{x}}{\lim\limits_{x\to 0}p^2x\cdot\lim\limits_{x\to 0}\dfrac{1-\cos px}{(px)^2}+\lim\limits_{x\to 0}\left(p\dfrac{\sin px}{px}\right)}=$$
$$=\frac{0\cdot\dfrac{1}{2}+1}{0\cdot\dfrac{1}{2}+p\cdot 1}=\underline{\frac{1}{p}}.$$

1.2.61 $\lim\limits_{x\to\pi}\dfrac{\sin mx}{\sin nx},\quad m,n\in Z.$

Řešení:

Můžeme udělat vhodnou úpravu $mx=(mx-m\pi)+m\pi$. Dostáváme

$$\frac{\sin mx}{\sin nx}=\frac{[\sin(mx-m\pi)+m\pi]}{\sin[(nx-n\pi)+n\pi]}=\frac{\sin m(x-\pi)\cos m\pi}{\sin n(x-\pi)\cos n\pi}=\frac{(-1)^m\sin m(x-\pi)}{(-1)^n\sin n(x-\pi)}=$$
$$=(-1)^{m-n}\frac{\sin m(x-\pi)}{\sin n(x-\pi)}\quad \textit{odtud}$$

$$\lim_{x\to\pi}\frac{\sin mx}{\sin nx}=\lim_{x\to\pi}\left[(-1)^{m-n}\frac{\sin m(x-\pi)}{\sin n(x-\pi)}\right]=(-1)^{m-n}\lim_{x\to\pi}\frac{m\cdot\dfrac{\sin m(x-\pi)}{m(x-\pi)}}{n\cdot\dfrac{\sin n(x-\pi)}{n(x-\pi)}}=$$

$$=(-1)^{m-n}\frac{m}{n}\cdot\frac{\lim\limits_{x\to\pi}\dfrac{\sin m(x-\pi)}{m(x-\pi)}}{\lim\limits_{x\to\pi}\dfrac{\sin n(x-\pi)}{n(x-\pi)}}$$

$\lim\limits_{x\to\pi}\dfrac{\sin m(x-\pi)}{m(x-\pi)}$ vypočteme snadno podle věty o limitě složené funkce. Vezmeme vnitřní funkci $g(x)=m(x-\pi)$ a vnější funkci $f(y)=\dfrac{\sin y}{y}$. Platí

$$\lim_{x\to\pi}m(x-\pi)=0,\quad \lim_{y\to 0}\frac{\sin y}{y}=1$$

Pro limitu složené funkce $f(g(x))=\dfrac{\sin m(x-\pi)}{m(x-\pi)}$ potom dostáváme

$$\lim_{x\to\pi}\frac{\sin m(x-\pi)}{m(x-\pi)}=\lim_{y\to 0}\frac{\sin y}{y}=1.$$

Tedy podle předchozího

$$\lim_{x\to\pi}\frac{\sin mx}{\sin nx}=(-1)^{m-n}\frac{m}{n}\cdot\frac{1}{1}=\underline{(-1)^{m-n}\frac{m}{n}}.$$

1.2.62 $\lim\limits_{x\to\frac{\pi}{4}} tg\,2x\cdot tg\left(\frac{\pi}{4}-x\right)$

Řešení:

Je vhodné nejprve tangenty vyjádřit pomocí sinu a kosinu.

$$tg\,2x\cdot tg\left(\frac{\pi}{4}-x\right)=\frac{\sin 2x}{\cos 2x}\cdot\frac{\sin\left(\frac{\pi}{4}-x\right)}{\cos\left(\frac{\pi}{4}-x\right)}$$

Nyní dáme zvlášť tu část limitované funkce, která je spojitá v bodě $\frac{\pi}{4}$.

$$\frac{\sin 2x}{\cos 2x}\cdot\frac{\sin\left(\frac{\pi}{4}-x\right)}{\cos\left(\frac{\pi}{4}-x\right)}=\frac{\sin 2x}{\cos\left(\frac{\pi}{4}-x\right)}\cdot\frac{\sin\left(\frac{\pi}{4}-x\right)}{\cos 2x}$$

Odtud

$$\lim_{x\to\frac{\pi}{4}} tg\,2x\cdot tg\left(\frac{\pi}{4}-x\right) = \lim_{x\to\frac{\pi}{4}}\left[\frac{\sin 2x}{\cos\left(\frac{\pi}{4}-x\right)}\cdot\frac{\sin\left(\frac{\pi}{4}-x\right)}{\cos 2x}\right] = \lim_{x\to\frac{\pi}{4}}\frac{\sin 2x}{\cos\left(\frac{\pi}{4}-x\right)}\cdot$$

$$\cdot\lim_{x\to\frac{\pi}{4}}\frac{\sin\left(\frac{\pi}{4}-x\right)}{\cos 2x} = \frac{\sin 2\cdot\frac{\pi}{4}}{\cos\left(\frac{\pi}{4}-\frac{\pi}{4}\right)}\cdot\lim_{x\to\frac{\pi}{4}}\frac{\sin\left(\frac{\pi}{4}-x\right)}{\cos 2x} = \lim_{x\to\frac{\pi}{4}}\frac{\sin\left(\frac{\pi}{4}-x\right)}{\cos 2x}$$

Celý výpočet bude v pořádku, pokud poslední limita existuje a je vlastní. Nejprve provedeme úpravu

$$\frac{\sin\left(\frac{\pi}{4}-x\right)}{\cos 2x} = \frac{\sin\left(\frac{\pi}{4}-x\right)}{\cos\left[2\left(x-\frac{\pi}{4}\right)+2\frac{\pi}{4}\right]} = \frac{\sin\left(\frac{\pi}{4}-x\right)}{-\sin 2\left(x-\frac{\pi}{4}\right)} = \frac{\sin\left(\frac{\pi}{4}-x\right)}{-2\sin\left(x-\frac{\pi}{4}\right)\cos\left(x-\frac{\pi}{4}\right)} =$$

$$= \frac{\sin\left(\frac{\pi}{4}-x\right)}{2\sin\left(\frac{\pi}{4}-x\right)\cos\left(x-\frac{\pi}{4}\right)} = \frac{1}{2\cos\left(x-\frac{\pi}{4}\right)}$$

Nyní už snadno dostáváme

$$\lim_{x\to\frac{\pi}{4}} tg\,2x\cdot tg\left(\frac{\pi}{4}-x\right) = \lim_{x\to\frac{\pi}{4}}\frac{1}{2\cos\left(x-\frac{\pi}{4}\right)} = \frac{1}{2\cos\left(\frac{\pi}{4}-\frac{\pi}{4}\right)} = \underline{\frac{1}{2}}$$

neboť funkce $\frac{1}{2\cos\left(x-\frac{\pi}{4}\right)}$ je v bodě $\frac{\pi}{4}$ spojitá.

1.2.63 $\lim_{x\to a}\frac{\sin x-\sin a}{x-a}$, $\lim_{x\to a}\frac{\cos x-\cos a}{x-a}$

Řešení:

Použijeme trigonometrické vzorce.

$$\lim_{x\to a}\frac{\sin x-\sin a}{x-a} = \lim_{x\to a}\frac{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}{x-a} = \lim_{x\to a}\left[\cos\frac{x+a}{2}\cdot\frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}}\right] =$$

$$= \lim_{x\to a}\cos\frac{x+a}{2}\cdot\lim_{x\to a}\frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}} = \cos a\cdot 1 = \underline{\cos a}.$$

$$\lim_{x \to a} \frac{\cos x - \cos a}{x - a} = \lim_{x \to a} \frac{-2\sin\frac{x+a}{2}\sin\frac{x-a}{2}}{x-a} = \lim_{x \to a}\left[-\sin\frac{x+a}{2}\cdot\frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}}\right] =$$

$$= -\lim_{x \to a}\sin\frac{x+a}{2}\cdot \lim_{x \to a}\frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}} = -\sin a \cdot 1 = \underline{-\sin a}.$$

1.2.64 $\lim_{x \to a} \frac{tg\,x - tg\,a}{x - a}$, $\lim_{x \to a} \frac{\cot g\,x - \cot g\,a}{x - a}$

Řešení:

$$\lim_{x \to a} \frac{tg\,x - tg\,a}{x - a} = \lim_{x \to a} \frac{\frac{\sin x}{\cos x} - \frac{\sin a}{\cos a}}{x - a} = \lim_{x \to a} \frac{\sin x \cos a - \cos x \sin a}{(x-a)\cos x \cos a} =$$

$$= \lim_{x \to a} \frac{\sin(x - a)}{(x-a)\cos x \cos a} = \lim_{x \to a} \frac{\sin(x - a)}{x - a}\cdot \lim_{x \to a} \frac{1}{\cos x \cos a} = 1 \cdot \frac{1}{\cos^2 a} = \underline{\frac{1}{\cos^2 a}}.$$

$$\lim_{x \to a} \frac{\cot g\,x - \cot g\,a}{x - a} = \lim_{x \to a} \frac{\frac{\cos x}{\sin x} - \frac{\cos a}{\sin a}}{x - a} = \lim_{x \to a} \frac{\cos x \sin a - \sin x \cos a}{(x-a)\sin x \sin a} =$$

$$= \lim_{x \to a} \frac{\sin(a - x)}{(x-a)\sin x \sin a} = -\lim_{x \to a} \frac{\sin(x - a)}{x - a}\cdot \lim_{x \to a} \frac{1}{\sin x \sin a} = -1 \cdot \frac{1}{\sin^2 a} = \underline{-\frac{1}{\sin^2 a}}.$$

1.2.65 $\lim_{x \to 0} \frac{\sin(a + 2x) - 2\sin(a + x) + \sin a}{x^2}$

Řešení:
Limitovanou funkci upravíme tak, aby argument u sinu se blížil nule pro x jdoucí k nule.

$$\frac{\sin(a + 2x) - 2\sin(a + x) + \sin a}{x^2} =$$

$$= \frac{\sin a \cos 2x + \cos a \sin 2x - 2\sin a \cos x - 2\cos a \sin x + \sin a}{x^2}.$$

$$\lim_{x \to 0} \frac{\sin(a + 2x) - 2\sin(a + x) + \sin a}{x^2} =$$

$$= \lim_{x \to 0} \frac{(\sin a \cos 2x - \sin a) + (2\sin a - 2\sin a \cos x) + \cos a \sin 2x - 2\cos a \sin x}{x^2} =$$

$$= \lim_{x \to 0} \frac{\sin a \cos 2x - \sin a}{x^2} + \lim_{x \to 0} \frac{2\sin a - 2\sin a \cos x}{x^2} + \lim_{x \to 0} \frac{\cos a \sin 2x - 2\cos a \sin x}{x^2} -$$

$$-4\sin a \lim_{x \to 0} \frac{1 - \cos 2x}{(2x)^2} + 2\sin a \lim_{x \to 0} \frac{1 - \cos x}{x^2} + \cos a \lim_{x \to 0} \frac{\sin 2x - 2\sin x}{x^2} = -4\sin a \frac{1}{2} +$$

$$+\,2\sin a\cdot\frac{1}{2}+\cos a\cdot\lim_{x\to 0}\frac{2\sin x\cos x-2\sin x}{x^2}=-\sin a+2\cos a\cdot\lim_{x\to 0}\frac{\sin x(\cos x-1)}{x^2}=$$

$$=-\sin a+2\cos a\cdot\lim_{x\to 0}\sin x\cdot\lim_{x\to 0}\frac{\cos x-1}{x^2}=-\sin a+2\cos a\cdot 0\cdot\left(-\frac{1}{2}\right)=\underline{-\sin a}.$$

Je ovšem možný ještě jiný postup.

$$\lim_{x\to 0}\frac{\sin(a+2x)-2\sin(a+x)+\sin a}{x^2}=\lim_{x\to 0}\frac{(\sin(a+2x)-\sin(a+x))+(\sin a-\sin(a+x))}{x^2}=$$

$$=\lim_{x\to 0}\frac{2\cos\frac{2a+3x}{2}\sin\frac{x}{2}+2\cos\frac{2a+x}{2}\sin\left(-\frac{x}{2}\right)}{x^2}=$$

$$=\lim_{x\to 0}\left[\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\frac{\cos\frac{2a+3x}{2}-\cos\frac{2a+x}{2}}{x}\right]=\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\lim_{x\to 0}\frac{\cos\frac{2a+3x}{2}-\cos\frac{2a+x}{2}}{x}=$$

$$=1\cdot\lim_{x\to 0}\frac{-2\sin(a+x)\sin\frac{x}{2}}{x}=-\lim_{x\to 0}\sin(a+x)\cdot\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}=-\sin a\cdot 1=\underline{-\sin a}.$$

1.2.66 $\lim_{x\to 0}\frac{\cot g(a+2x)-2\cot g(a+x)+\cot g\,a}{x^2}$

<u>Řešení:</u>

$$\lim_{x\to 0}\frac{\cot g(a+2x)-2\cot g(a+x)+\cot g\,a}{x^2}=\lim_{x\to 0}\frac{\frac{\cos(a+2x)}{\sin(a+2x)}-2\frac{\cos(a+x)}{\sin(a+x)}+\frac{\cos a}{\sin a}}{x^2}=$$

$$=\lim_{x\to 0}\frac{\cos(a+2x)\sin(a+x)\sin a-2\cos(a+x)\sin(a+2x)\sin a+\cos a\sin(a+2x)\sin(a+x)}{\sin(a+2x)\sin(a+x)\sin a\cdot x^2}=$$

$$=\lim_{x\to 0}\frac{1}{\sin(a+2x)\sin(a+x)\sin a}\cdot$$

$$\cdot\lim_{x\to 0}\frac{\cos(a+2x)\sin(a+x)\sin a-2\cos(a+x)\sin(a+2x)\sin a+\cos a\sin(a+2x)\sin(a+x)}{x^2}$$

První limita je zřejmě rovna $\frac{1}{\sin^3 a}$. Výraz v čitateli u druhé limity je dosti složitý, a proto si ho nejprve upravíme.

$$\cos(a+2x)\sin(a+x)\sin a-2\cos(a+x)\sin(a+2x)\sin a+\cos a\sin(a+2x)\sin(a+x)=$$

$$=\sin a(\cos(a+2x)\sin(a+x)-\cos(a+x)\sin(a+2x))+\sin(a+2x)(\cos a\sin(a+2x)-$$

$$-\cos(a+x)\sin a)=\sin a\sin(-x)+\sin(a+2x)\sin x=\sin x(\sin(a+2x)-\sin a)=$$

$$=\sin x\cdot 2\cos(a+x)\sin x=2\sin^2 x\cos(a+x).$$

Tedy

$$\lim_{x\to 0}\frac{\cot g\,(a+2x)-2\cot g(a+x)+\cot g\,a}{x^2}=\frac{1}{\sin^3 a}\cdot\lim_{x\to 0}\frac{2\sin^2 x\cos(a+x)}{x^2}=$$

$$=\frac{2}{\sin^3 a}\cdot\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\lim_{x\to 0}\cos(a+x)=\frac{2}{\sin^3 a}\cdot 1\cdot\cos a=\frac{2\cos a}{\sin^3 a}$$

Zde je ovšem možné použít též postup z příkladů 46, 50, 67

$$\lim_{x\to 0}\frac{\cot g\,(a+2x)-2\cot g(a+x)+\cot g\,a}{x^2}=$$

$$=\lim_{x\to 0}\frac{(\cot g\,(a+2x)-\cot g(a+x))+(\cot g\,a-\cot g(a+x))}{x^2}=$$

$$=\lim_{x\to 0}\frac{\left(\frac{\cos(a+2x)}{\sin(a+2x)}-\frac{\cos(a+x)}{\sin(a+x)}\right)+\left(\frac{\cos a}{\sin a}-\frac{\cos(a+x)}{\sin(a+x)}\right)}{x^2}=atd.,$$

který se od předchozího téměř neliší. Má snad jen tu výhodu, že při uvedení výrazu v závorkách na společného jmenovatele okamžitě vidíme příslušnou součtovou trigonometrickou formuli, protože výrazy nejsou komplikovány třetím faktorem jako výše, který bylo třeba vytýkat.

1.2.67 $\lim_{x\to 0}\frac{\sin(a+x)\sin(a+2x)-\sin^2 a}{x}$

Řešení:

Nejdříve upravíme čitatele zlomku. Můžeme použít trigonometrické formule.

$$\sin(a+x)\sin(a+2x)-\sin^2 a=(\sin a\,\cos x+\cos a\,\sin x)(\sin a\,\cos 2x+\cos a\,\sin 2x)-$$

$$-\sin^2 a=\sin^2 a\cos x\cos 2x+\sin a\cos a\sin 2x\cos x+\sin a\cos a\sin x\cos 2x+$$

$$+\cos^2 a\sin x\sin 2x-\sin^2 a$$

Odtud

$$\lim_{x\to 0}\frac{\sin(a+x)\sin(a+2x)-\sin^2 a}{x}=\sin^2 a\cdot\lim_{x\to 0}\frac{\cos x\cos 2x-1}{x}+$$

$$+\sin a\cos a\cdot\lim_{x\to 0}\frac{\sin 2x}{x}\cdot\cos x+\sin a\cos a\cdot\lim_{x\to 0}\frac{\sin x}{x}\cdot\cos 2x++\cos^2 a\cdot\lim_{x\to 0}\frac{\sin x}{x}\cdot\sin 2x=$$

$$=-\sin^2 a\cdot\lim_{x\to 0}\frac{1-\cos x\cos 2x}{x}+2\sin a\cos a+\sin a\cos a+\cos^2 a\cdot 0=$$

$$=-\sin^2 a\cdot\lim_{x\to 0}\frac{1-\cos x\cos 2x}{x}+3\sin a\cos a$$

Zbývá tedy vypočíst ještě $\lim_{x\to 0}\frac{1-\cos x\cos 2x}{x}$.

$$\lim_{x\to 0}\frac{1-\cos x\cos 2x}{x}=\lim_{x\to 0}\frac{\cos^2 x+\sin^2 x-\cos x(\cos^2 x-\sin^2 x)}{x}=$$

$$=\lim_{x\to 0}\frac{\cos^2 x(1-\cos x)+\sin^2 x(1+\cos x)}{x}=\lim_{x\to 0}\cos^2 x\cdot\frac{1-\cos x}{x^2}\cdot x+$$

$$+\lim_{x\to 0}\frac{\sin x}{x}\cdot\sin x\cdot(1+\cos x)=1\cdot\frac{1}{2}\cdot 0+1\cdot 0\cdot 2=0$$

Celkem tak dostáváme

$$\lim_{x\to 0}\frac{\sin(a+x)\sin(a+2x)-\sin^2 a}{x} = -\sin^2 a\cdot 0 + 3\sin a\cos a = 3\sin a\cos a = \underline{\frac{3}{2}\sin 2a}.$$

Můžeme ještě ukázat jeden, poněkud obratnější způsob výpočtu této limity. Nejprve upravíme čitatele limitované funkce.

$$\sin(a+x)\sin(a+2x)-\sin^2 a = \left(\sin(a+x)\sin(a+2x)-\sin(a+x)\sin a\right)+$$
$$+\left(\sin(a+x)\sin a-\sin^2 a\right) = \sin(a+x)\left(\sin(a+2x)-\sin a\right)+\sin a\left(\sin(a+x)-\sin a\right)=$$
$$=\sin(a+x)\cdot 2\cos(a+x)\sin x + \sin a\cdot 2\cos\frac{2a+x}{2}\cdot\sin\frac{x}{2}$$

Odtud

$$\lim_{x\to 0}\frac{\sin(a+x)\sin(a+2x)-\sin^2 a}{x} = \lim_{x\to 0}\left[\sin(a+x)\cos(a+x)\frac{\sin x}{x}\right]+$$
$$+\sin a\cdot\lim_{x\to 0}\left[\cos\frac{2a+x}{2}\cdot\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right] = 2\sin a\cos a + \sin a\cos a = \underline{\frac{3}{2}\sin 2a}.$$

1.2.68 $\lim_{x\to 0}\frac{1-\cos x\cos 2x\cos 3x}{1-\cos x}$

__Řešení:__

Použijeme metodu vhodného přičtení a odečtení

$$1-\cos x\cos 2x\cos 3x = \left(1-\cos 2x\cos 3x\right)+\left(\cos 2x\cos 3x-\cos x\cos 2x\cos 3x\right)=$$
$$=\left(1-\cos 3x\right)+\left(\cos 3x-\cos 2x\cos 3x\right)+\cos 2x\cos 3x\left(1-\cos x\right)=\left(1-\cos 3x\right)+$$
$$+\cos 3x\left(1-\cos 2x\right)+\cos 2x\cos 3x\left(1-\cos x\right)$$

Odtud

$$\lim_{x\to 0}\frac{1-\cos x\cos 2x\cos 3x}{1-\cos x} = \lim_{x\to 0}\frac{1-\cos 3x}{1-\cos x}+\lim_{x\to 0}\left[\cos 3x\cdot\frac{1-\cos 2x}{1-\cos x}\right]+\lim_{x\to 0}\cos 2x\cos 3x =$$
$$=\lim_{x\to 0}\frac{9\frac{1-\cos 3x}{(3x)^2}}{\frac{1-\cos x}{x^2}}+\lim_{x\to 0}\left[\cos 3x\cdot\frac{4\frac{1-\cos 2x}{(2x)^2}}{\frac{1-\cos x}{x^2}}\right]+1 = \frac{9\cdot\frac{1}{2}}{\frac{1}{2}}+1\cdot\frac{4\cdot\frac{1}{2}}{\frac{1}{2}}+1 = 9+4+1 = \underline{14}.$$

1.2.69 $\lim_{x\to 0}\frac{tg(a+x)\ tg(a-x)-tg^2 a}{x^2}$

__Řešení:__

$$tg(a+x)\ tg(a-x)-tg^2 a = \frac{tg\,a+tg\,x}{1-tg\,a\,tg\,x}\cdot\frac{tg\,a-tg\,x}{1+tg\,a\,tg\,x}-tg^2 a =$$
$$=\frac{tg^2 a-tg^2 x-tg^2 a\left(1-tg^2 a\ tg^2 x\right)}{1-tg^2 a\ tg^2 x}=\frac{tg^2 x\left(tg^4 a-1\right)}{1-tg^2 a\ tg^2 x}$$

Odtud

$$\lim_{x\to 0}\frac{tg\left(a+x\right)\,tg\left(a-x\right)-tg^2 a}{x^2}=\left(tg^4 a-1\right)\cdot\lim_{x\to 0}\frac{tg^2 x}{x^2}\cdot\lim_{x\to 0}\frac{1}{1-tg^2 a\,tg^2 x}=\left(tg^4 a-1\right)\cdot 1\cdot\frac{1}{1}=$$

$$=tg^4 a-1=\frac{\sin^4 a}{\cos^4 a}-1=\frac{\sin^4 a-\cos^4 a}{\cos^4 a}=\frac{\left(\sin^2 a-\cos^2 a\right)\left(\sin^2 a+\cos^2 a\right)}{\cos^4 a}=\underline{-\frac{\cos 2a}{\cos^4 a}}.$$

1.2.70 $\lim_{x\to 0}\frac{1-\cos\left(1-\cos x\right)}{x^4}$

Řešení:

$$\lim_{x\to 0}\frac{1-\cos\left(1-\cos x\right)}{x^4}=\lim_{x\to 0}\left[\frac{1-\cos\left(1-\cos x\right)}{\left(1-\cos x\right)^2}\cdot\frac{\left(1-\cos x\right)^2}{x^4}\right]=\lim_{x\to 0}\frac{1-\cos\left(1-\cos x\right)}{\left(1-\cos x\right)^2}\cdot$$

$$\cdot\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)^2=\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2=\underline{\frac{1}{8}}.$$

1.2.71 $\lim_{x\to\frac{\pi}{6}}\frac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}$

Řešení:

Dosadíme-li $\frac{\pi}{6}$ do čitatele a do jmenovatele uvažovaného zlomku, vyjde nám vždy 0. Znamená to tedy, že číslo $\sin\frac{\pi}{6}=\frac{1}{2}$ je kořenem jak polynomu $2y^2+y-1$, tak i polynomu $2y^2-3y+1$. Snadno dostáváme rozklady

$$2y^2+y-1=2\left(y+1\right)\left(y-\frac{1}{2}\right)$$

$$2y^2-3y+1=2\left(y-\frac{1}{2}\right)\left(y-1\right)$$

Odtud

$$\lim_{x\to\frac{\pi}{6}}\frac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}=\lim_{x\to\frac{\pi}{6}}\frac{2\left(\sin x+1\right)\left(\sin x-\frac{1}{2}\right)}{2\left(\sin x-\frac{1}{2}\right)\left(\sin x-1\right)}=\lim_{x\to\frac{\pi}{6}}\frac{\sin x+1}{\sin x-1}=\frac{\frac{1}{2}+1}{\frac{1}{2}-1}=\underline{-3}.$$

1.2.72 $\lim\limits_{x \to \frac{\pi}{3}} \dfrac{\sin\left(x - \frac{\pi}{3}\right)}{1 - 2\cos x}$

Řešení:

$$\lim_{x \to \frac{\pi}{3}} \frac{\sin\left(x - \frac{\pi}{3}\right)}{1 - 2\cos x} = \lim_{x \to \frac{\pi}{3}} \frac{\sin\left(x - \frac{\pi}{3}\right)}{1 - 2\cos\left[\left(x - \frac{\pi}{3}\right) + \frac{\pi}{3}\right]} =$$

$$= \lim_{x \to \frac{\pi}{3}} \frac{\sin\left(x - \frac{\pi}{3}\right)}{1 - 2\left(\cos\left(x - \frac{\pi}{3}\right)\cdot\frac{1}{2} - \sin\left(x - \frac{\pi}{3}\right)\cdot\frac{\sqrt{3}}{2}\right)} =$$

$$= \lim_{x \to \frac{\pi}{3}} \frac{\sin\left(x - \frac{\pi}{3}\right)}{\left(1 - \cos\left(x - \frac{\pi}{3}\right)\right) + \sqrt{3}\sin\left(x - \frac{\pi}{3}\right)} =$$

$$= \lim_{x \to \frac{\pi}{3}} \frac{\dfrac{\sin\left(x - \frac{\pi}{3}\right)}{x - \frac{\pi}{3}}}{\dfrac{1 - \cos\left(x - \frac{\pi}{3}\right)}{\left(x - \frac{\pi}{3}\right)^2}\cdot\left(x - \frac{\pi}{3}\right) + \sqrt{3}\,\dfrac{\sin\left(x - \frac{\pi}{3}\right)}{x - \frac{\pi}{3}}} = \frac{1}{\frac{1}{2}\cdot 0 + \sqrt{3}} = \frac{1}{\sqrt{3}} = \underline{\frac{\sqrt{3}}{3}}.$$

1.2.73 $\lim\limits_{x \to \frac{\pi}{3}} \dfrac{tg^3 x - 3\,tg\,x}{\cos\left(x + \frac{\pi}{6}\right)}$

Řešení:

$$\lim_{x \to \frac{\pi}{3}} \frac{tg^3 x - 3\,tg\,x}{\cos\left(x + \frac{\pi}{6}\right)} = \lim_{x \to \frac{\pi}{3}} \frac{tg\,x\left(tg\,x - \sqrt{3}\right)\left(tg\,x + \sqrt{3}\right)}{\cos\left(x + \frac{\pi}{6}\right)} = \lim_{x \to \frac{\pi}{3}} tg\,x \cdot \lim_{x \to \frac{\pi}{3}} \frac{tg\,x - \sqrt{3}}{\cos\left(x + \frac{\pi}{6}\right)}$$

$$\cdot \lim_{x \to \frac{\pi}{3}} \left(tg\,x + \sqrt{3}\right) = \sqrt{3}\cdot \lim_{x \to \frac{\pi}{3}} \frac{tg\,x - \sqrt{3}}{\cos\left(x + \frac{\pi}{6}\right)}\cdot 2\sqrt{3} = 6\lim_{x \to \frac{\pi}{3}} \frac{\frac{\sin x}{\cos x} - \sqrt{3}}{\cos\left(x + \frac{\pi}{6}\right)} =$$

$$= 6\lim_{x\to\frac{\pi}{3}} \frac{\sin x - \sqrt{3}\cos x}{\cos x \cos\left(x+\frac{\pi}{6}\right)} = 6\lim_{x\to\frac{\pi}{3}} \frac{1}{\cos x}\cdot \lim_{x\to\frac{\pi}{3}} \frac{\sin x - \sqrt{3}\cos x}{\cos\left(x+\frac{\pi}{6}\right)} =$$

$$= 6\cdot\frac{1}{\frac{1}{2}}\cdot 2\lim_{x\to\frac{\pi}{3}} \frac{\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x}{\cos\left[\left(x-\frac{\pi}{3}\right)+\frac{\pi}{2}\right]} = 24\lim_{x\to\frac{\pi}{3}} \frac{\sin x\cos\frac{\pi}{3} - \cos x \sin\frac{\pi}{3}}{-\sin\left(x-\frac{\pi}{3}\right)} =$$

$$= -24\lim_{x\to\frac{\pi}{3}} \frac{\sin\left(x-\frac{\pi}{3}\right)}{\sin\left(x-\frac{\pi}{3}\right)} = \underline{-24}.$$

1.2.74 $\lim\limits_{x\to 2} \dfrac{x^2-4}{\cos\frac{\pi}{4}x}$

<u>Řešení:</u>

$$\lim_{x\to 2} \frac{x^2-4}{\cos\frac{\pi}{4}x} = \lim_{x\to 2} \frac{(x-2)(x+2)}{\cos\frac{\pi}{4}[(x-2)+2]} = \lim_{x\to 2} (x+2)\cdot\frac{x-2}{\cos\left[\frac{\pi}{4}(x-2)+\frac{\pi}{2}\right]} =$$

$$= 4\cdot\lim_{x\to 2} \frac{x-2}{-\sin\frac{\pi}{4}(x-2)} = -4\cdot\frac{4}{\pi}\cdot\lim_{x\to 2} \frac{\frac{\pi}{4}(x-2)}{\sin\frac{\pi}{4}(x-2)} = \underline{-\frac{16}{\pi}}.$$

1.2.75 $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\cos x - \sin x}{\cos 2x}$

<u>Řešení:</u>

$$\lim_{x\to\frac{\pi}{4}} \frac{\cos x - \sin x}{\cos 2x} = \lim_{x\to\frac{\pi}{4}} \frac{\cos x - \sin x}{\cos^2 x - \sin^2 x} = \lim_{x\to\frac{\pi}{4}} \frac{\cos x - \sin x}{(\cos x - \sin x)(\cos x + \sin x)} =$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{1}{\cos x + \sin x} = \frac{1}{\cos\frac{\pi}{4} + \sin\frac{\pi}{4}} = \frac{1}{\sqrt{2}} = \underline{\frac{\sqrt{2}}{2}}.$$

1.2.76 $\lim\limits_{x \to \frac{\pi}{6}} \dfrac{\sin\left(x - \frac{\pi}{6}\right)}{\frac{\sqrt{3}}{2} - \cos x}$

Řešení:

$$\lim_{x \to \frac{\pi}{6}} \frac{\sin\left(x - \frac{\pi}{6}\right)}{\frac{\sqrt{3}}{2} - \cos x} = \lim_{x \to \frac{\pi}{6}} \frac{\sin\left(x - \frac{\pi}{6}\right)}{\cos\frac{\pi}{6} - \cos x} = \lim_{x \to \frac{\pi}{6}} \frac{\sin\left(x - \frac{\pi}{6}\right)}{-2\sin\frac{1}{2}\left(\frac{\pi}{6} + x\right)\sin\frac{1}{2}\left(\frac{\pi}{6} - x\right)} =$$

$$= -\frac{1}{2} \cdot \lim_{x \to \frac{\pi}{6}} \frac{1}{\sin\frac{1}{2}\left(\frac{\pi}{6} + x\right)} \cdot \lim_{x \to \frac{\pi}{6}} \frac{\sin\left(x - \frac{\pi}{6}\right)}{\sin\frac{1}{2}\left(\frac{\pi}{6} - x\right)} =$$

$$= -\frac{1}{2} \cdot \frac{1}{\frac{1}{2}} \cdot \lim_{x \to \frac{\pi}{6}} \frac{2\sin\frac{1}{2}\left(x - \frac{\pi}{6}\right)\cos\frac{1}{2}\left(x - \frac{\pi}{6}\right)}{\sin\frac{1}{2}\left(\frac{\pi}{6} - x\right)} = 2\lim_{x \to \frac{\pi}{6}} \cos\frac{1}{2}\left(x - \frac{\pi}{6}\right) = \underline{2}.$$

Byl ovšem taky možný jiný postup výpočtu poslední limity

$$\lim_{x \to \frac{\pi}{6}} \frac{\sin\left(x - \frac{\pi}{6}\right)}{\sin\frac{1}{2}\left(\frac{\pi}{6} - x\right)} = -\lim_{x \to \frac{\pi}{6}} \frac{\sin\left(x - \frac{\pi}{6}\right)}{\sin\frac{1}{2}\left(x - \frac{\pi}{6}\right)} = -\lim_{x \to \frac{\pi}{6}} \frac{\frac{\sin\left(x - \frac{\pi}{6}\right)}{x - \frac{\pi}{6}}}{\frac{1}{2} \cdot \frac{\sin\frac{1}{2}\left(x - \frac{\pi}{6}\right)}{\frac{1}{2}\left(x - \frac{\pi}{6}\right)}} = -\frac{1}{\frac{1}{2} \cdot 1} = 2.$$

což dosazeno výše, dává opět výsledek 2.

1.2.77 $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{1 - \cot g^3 x}{2 - \cot g\, x - \cot g^3 x}$

Řešení:

$$\lim_{x \to \frac{\pi}{4}} \frac{1 - \cot g^3 x}{2 - \cot g\, x - \cot g^3 x} = \lim_{x \to \frac{\pi}{4}} \frac{1 - \cot g^3 x}{(1 - \cot g\, x) + (1 - \cot g^3 x)} =$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{(1 - \cot g\, x)(1 + \cot g\, x + \cot g^2 x)}{(1 - \cot g\, x) + (1 - \cot g x)(1 + \cot g\, x + \cot g^2 x)} = \lim_{x \to \frac{\pi}{4}} \frac{1 + \cot g\, x + \cot g^2 x}{1 + 1 + \cot g\, x + \cot g^2 x} =$$

$$= \frac{1 + 1 + 1}{1 + 1 + 1 + 1} = \underline{\frac{3}{4}}.$$

1.2.78 $\lim\limits_{x\to 0} \dfrac{\sqrt{1+tg\,x}-\sqrt{1+\sin x}}{x^3}$

<u>*Řešení:*</u>

$$\lim_{x\to 0} \frac{\sqrt{1+tg\,x}-\sqrt{1+\sin x}}{x^3} = \lim_{x\to 0} \frac{(1+tg\,x)-(1+\sin x)}{x^3\left(\sqrt{1+tg\,x}+\sqrt{1+\sin x}\right)} =$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{1+tg\,x}+\sqrt{1+\sin x}} \cdot \lim_{x\to 0} \frac{tg\,x-\sin x}{x^3} = \frac{1}{2}\cdot \lim_{x\to 0} \frac{\frac{\sin x}{\cos x}-\sin x}{x^3} = \frac{1}{2}\cdot \lim_{x\to 0} \frac{\sin x(1-\cos x)}{x^3\cos x} =$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{1}{\cos x}\cdot \lim_{x\to 0}\frac{\sin x}{x}\cdot \lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}\cdot 1\cdot 1\cdot \frac{1}{2} = \underline{\frac{1}{4}}.$$

1.2.79 $\lim\limits_{x\to 0} \dfrac{x^2}{\sqrt{1+x\sin x}-\sqrt{\cos x}}$

<u>*Řešení:*</u>

$$\lim_{x\to 0} \frac{x^2}{\sqrt{1+x\sin x}-\sqrt{\cos x}} = \lim_{x\to 0} \frac{x^2\left(\sqrt{1+x\sin x}+\sqrt{\cos x}\right)}{1+x\sin x-\cos x} = \lim_{x\to 0}\left(\sqrt{1+x\sin x}+\sqrt{\cos x}\right)\cdot$$

$$\cdot \lim_{x\to 0} \frac{x^2}{1-\cos x+x\sin x} = 2\cdot \lim_{x\to 0} \frac{1}{\frac{1-\cos x}{x^2}+\frac{\sin x}{x}} = 2\cdot \frac{1}{\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}+\lim\limits_{x\to 0}\frac{\sin x}{x}} =$$

$$= 2\cdot \frac{1}{\frac{1}{2}+1} = \underline{\frac{4}{3}}.$$

1.2.80 $\lim\limits_{x\to 0} \dfrac{\sqrt{\cos x}-\sqrt[3]{\cos x}}{\sin^2 x}$

<u>*Řešení:*</u>

$$\lim_{x\to 0} \frac{\sqrt{\cos x}-\sqrt[3]{\cos x}}{\sin^2 x} = \lim_{x\to 0} \frac{\left(\sqrt{\cos x}-1\right)+\left(1-\sqrt[3]{\cos x}\right)}{\sin^2 x} = \lim_{x\to 0}\frac{\sqrt{\cos x}-1}{\sin^2 x} + \lim_{x\to 0}\frac{1-\sqrt[3]{\cos x}}{\sin^2 x}$$

$$\lim_{x\to 0}\frac{\sqrt{\cos x}-1}{\sin^2 x} = \lim_{x\to 0}\frac{\cos x-1}{\sin^2 x\left(\sqrt{\cos x}+1\right)} = \lim_{x\to 0}\frac{1}{\sqrt{\cos x}+1}\cdot \lim_{x\to 0}\frac{\cos x-1}{\sin^2 x} =$$

$$= \frac{1}{2}\lim_{x\to 0}\left[\frac{\cos x-1}{x^2}\cdot\frac{x^2}{\sin^2 x}\right] = \frac{1}{2}\lim_{x\to 0}\frac{\cos x-1}{x^2}\cdot \lim_{x\to 0}\left(\frac{x}{\sin x}\right)^2 = \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot 1 = -\frac{1}{4}$$

$$\lim_{x\to 0}\frac{1-\sqrt[3]{\cos x}}{\sin^2 x} = \lim_{x\to 0}\frac{1-\cos x}{\left(1+\sqrt[3]{\cos x}+\left(\sqrt[3]{\cos x}\right)^2\right)\sin^2 x} = \lim_{x\to 0}\frac{1}{1+\sqrt[3]{\cos x}+\left(\sqrt[3]{\cos x}\right)^2}\cdot$$

$$\cdot \lim_{x\to 0}\frac{1-\cos x}{\sin^2 x} = \frac{1}{3}\cdot \lim_{x\to 0}\left[\frac{1-\cos x}{x^2}\cdot\frac{x^2}{\sin^2 x}\right] = \frac{1}{3}\cdot \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot \lim_{x\to 0}\left(\frac{x}{\sin x}\right)^2 = \frac{1}{3}\cdot\frac{1}{2}\cdot 1 = \frac{1}{6}.$$

Celkem tedy dostáváme

$$\lim_{x\to 0}\frac{\sqrt{\cos x}-\sqrt[3]{\cos x}}{\sin^2 x}=-\frac{1}{4}+\frac{1}{6}=\underline{-\frac{1}{12}}.$$

1.2.81 $\lim_{x\to 0+}\frac{1-\sqrt{\cos x}}{1-\cos\left(\sqrt{x}\right)}$

Řešení:

$$\lim_{x\to 0+}\frac{1-\sqrt{\cos x}}{1-\cos\left(\sqrt{x}\right)}=\lim_{x\to 0+}\frac{1-\cos x}{\left(1-\cos\left(\sqrt{x}\right)\right)\cdot\left(1+\sqrt{\cos x}\right)}=\lim_{x\to 0+}\frac{1}{1+\sqrt{\cos x}}\cdot$$

$$\cdot\lim_{x\to 0+}\frac{1-\cos x}{1-\cos\left(\sqrt{x}\right)}=\frac{1}{2}\lim_{x\to 0+}\left[\frac{1-\cos x}{x^2}\cdot x\cdot\frac{\left(\sqrt{x}\right)^2}{1-\cos\left(\sqrt{x}\right)}\right]=\frac{1}{2}\lim_{x\to 0+}\frac{1-\cos x}{x^2}\cdot\lim_{x\to 0+}x\cdot$$

$$\cdot\lim_{x\to 0+}\frac{\left(\sqrt{x}\right)^2}{1-\cos\left(\sqrt{x}\right)}=\frac{1}{2}\cdot\frac{1}{2}\cdot 0\cdot 2=\underline{0}.$$

1.2.82 $\lim_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}\ \sqrt[3]{\cos 3x}}{x^2}$

Řešení:

Nejprve upravíme čitatele.

$$1-\cos x\ \sqrt{\cos 2x}\ \sqrt[3]{\cos 3x}=\left(1-\sqrt{\cos 2x}\ \sqrt[3]{\cos 3x}\right)+$$

$$+\left(\sqrt{\cos 2x}\ \sqrt[3]{\cos 3x}-\cos x\sqrt{\cos 2x}\ \sqrt[3]{\cos 3x}\right)=\left(1-\sqrt[3]{\cos 3x}\right)+\left(\sqrt[3]{\cos 3x}-\sqrt{\cos 2x}\sqrt[3]{\cos 3x}\right)+$$

$$+\left(\sqrt{\cos 2x}\ \sqrt[3]{\cos 3x}-\cos x\sqrt{\cos 2x}\ \sqrt[3]{\cos 3x}\right)$$

$$\lim_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}\ \sqrt[3]{\cos 3x}}{x^2}=\lim_{x\to 0}\frac{1-\sqrt[3]{\cos 3x}}{x^2}+\lim_{x\to 0}\frac{\sqrt[3]{\cos 3x}-\sqrt{\cos 2x}\sqrt[3]{\cos 3x}}{x^2}+$$

$$+\lim_{x\to 0}\frac{\sqrt{\cos 2x}\ \sqrt[3]{\cos 3x}-\cos x\sqrt{\cos 2x}\ \sqrt[3]{\cos 3x}}{x^2}$$

$$\lim_{x\to 0}\frac{1-\sqrt[3]{\cos 3x}}{x^2}=\lim_{x\to 0}\frac{1-\cos 3x}{x^2\left(1+\sqrt[3]{\cos 3x}+\left(\sqrt[3]{\cos 3x}\right)^2\right)}=\lim_{x\to 0}\frac{1-\cos 3x}{x^2}\cdot$$

$$\cdot\lim_{x\to 0}\frac{1}{1+\sqrt[3]{\cos 3x}+\left(\sqrt[3]{\cos 3x}\right)^2}=\frac{9}{2}\cdot\frac{1}{3}=\frac{3}{2}$$

$$\lim_{x\to 0}\frac{\sqrt[3]{\cos 3x}-\sqrt{\cos 2x}\sqrt[3]{\cos 3x}}{x^2}=\lim_{x\to 0}\sqrt[3]{\cos 3x}\cdot\lim_{x\to 0}\frac{1-\sqrt{\cos 2x}}{x^2}=1\cdot\lim_{x\to 0}\frac{1-\cos 2x}{x^2\left(1+\sqrt{\cos 2x}\right)}=$$

$$=\lim_{x\to 0}\frac{1-\cos 2x}{x^2}\cdot\lim_{x\to 0}\frac{1}{1+\sqrt{\cos 2x}}=\frac{4}{2}\cdot\frac{1}{2}=1$$

$$\lim_{x\to 0}\frac{\sqrt{\cos 2x}\,\sqrt[3]{\cos 3x}-\cos x\sqrt{\cos 2x}\,\sqrt[3]{\cos 3x}}{x^2}=\lim_{x\to 0}\left[\sqrt{\cos 2x}\,\sqrt[3]{\cos 3x}\right]\cdot\lim_{x\to 0}\frac{1-\cos x}{x^2}=$$

$$=1\cdot\frac{1}{2}=\frac{1}{2}$$

Dohromady tak dostáváme

$$\lim_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}\,\sqrt[3]{\cos 3x}}{x^2}=\frac{3}{2}+1+\frac{1}{2}=\underline{3}.$$

1.2.83 $\lim\limits_{x\to\infty}\left(\sin\sqrt{x+1}-\sin\sqrt{x}\right)$

Řešení:

Použijeme trigonometrickou formuli pro $\sin\alpha-\sin\beta$.

$$\lim_{x\to\infty}\left(\sin\sqrt{x+1}-\sin\sqrt{x}\right)=\lim_{x\to\infty}2\cos\frac{\sqrt{x+1}+\sqrt{x}}{2}\cdot\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}$$

Funkce $\cos\frac{\sqrt{x+1}+\sqrt{x}}{2}$ je na celém svém definičním oboru omezená. Dále pak

$$\lim_{x\to\infty}\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}=\lim_{x\to\infty}\sin\frac{x+1-x}{2\left(\sqrt{x+1}+\sqrt{x}\right)}=\lim_{x\to\infty}\sin\frac{1}{2\left(\sqrt{x+1}+\sqrt{x}\right)}=0$$

$$\lim_{x\to\infty}\left(\sin\sqrt{x+1}-\sin\sqrt{x}\right)=\underline{0}.$$

1.2.84 $\lim\limits_{x\to 0}\frac{tg\,(tg\,x)-\sin(\sin x)}{tg\,x-\sin x}$

Řešení:

$$\lim_{x\to 0}\frac{tg\,(tg\,x)-\sin(\sin x)}{tg\,x-\sin x}=\lim_{x\to 0}\frac{tg\,(tg\,x)-\sin(\sin x)}{\frac{\sin x}{\cos x}-\sin x}=\lim_{x\to 0}\frac{tg\,(tg\,x)-\sin(\sin x)}{\frac{\sin x(1-\cos x)}{\cos x}}=$$

$$=\lim_{x\to 0}\left[\cos x\cdot\frac{tg\,(tg\,x)-\sin(\sin x)}{\sin x(1-\cos x)}\right]=\lim_{x\to 0}\cos x\cdot\lim_{x\to 0}\frac{tg\,(tg\,x)-\sin(\sin x)}{\sin x(1-\cos x)}=$$

$$=1\cdot\lim_{x\to 0}\frac{\frac{tg\,(tg\,x)-\sin(\sin x)}{x^3}}{\frac{\sin x}{x}\cdot\frac{1-\cos x}{x^2}}=\frac{\lim\limits_{x\to 0}\frac{tg\,(tg\,x)-\sin(\sin x)}{x^3}}{\lim\limits_{x\to 0}\frac{\sin x}{x}\cdot\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}}=2\lim_{x\to 0}\frac{tg\,(tg\,x)-\sin(\sin x)}{x^3}=$$

$$=2\lim_{x\to 0}\frac{[tg\,(tg\,x)-\sin(tg\,x)]+[\sin(tg\,x)-\sin(\sin x)]}{x^3}=2\lim_{x\to 0}\frac{tg\,(tg\,x)-\sin(tg\,x)}{x^3}+$$

$$+2\lim_{x\to 0}\frac{\sin(tg\,x)-\sin(\sin x)}{x^3}$$

Vypočteme nyní postupně obě poslední limity

$$\lim_{x\to 0}\frac{tg(tg\,x)-\sin(tg\,x)}{x^3}=\lim_{x\to 0}\frac{\sin(tg\,x)-\sin(tg\,x)\cos(tg\,x)}{x^3\cos(tg\,x)}=$$

$$=\lim_{x\to 0}\left[\frac{\sin(tg\,x)}{x}\cdot\frac{1-\cos(tg\,x)}{x^2}\cdot\frac{1}{\cos(tg\,x)}\right]=\lim_{x\to 0}\left[\frac{\sin(tg\,x)}{tg\,x}\cdot\frac{tg\,x}{x}\cdot\frac{1-\cos(tg\,x)}{tg^2x}\cdot\left(\frac{tg\,x}{x}\right)^2\cdot\right.$$

$$\left.\cdot\frac{1}{\cos(tg\,x)}\right]=\lim_{x\to 0}\frac{\sin(tg\,x)}{tg\,x}\cdot\lim_{x\to 0}\frac{tg\,x}{x}\cdot\lim_{x\to 0}\frac{1-\cos(tg\,x)}{tg^2x}\cdot\left(\lim_{x\to 0}\frac{tg\,x}{x}\right)^2\cdot\lim_{x\to 0}\frac{1}{\cos(tg\,x)}=$$

$$=1\cdot 1\cdot\frac{1}{2}\cdot 1^2\cdot\frac{1}{1}=\frac{1}{2}.$$

$$\lim_{x\to 0}\frac{\sin(tg\,x)-\sin(\sin x)}{x^3}=\lim_{x\to 0}\frac{2\cos\left(\frac{tg\,x+\sin x}{2}\right)\sin\left(\frac{tg\,x-\sin x}{2}\right)}{x^3}=2\lim_{x\to 0}\cos\left(\frac{tg\,x+\sin x}{2}\right)\cdot$$

$$\cdot\lim_{x\to 0}\frac{\sin\left(\frac{tg\,x-\sin x}{2}\right)}{x^3}=2\cdot 1\cdot\lim_{x\to 0}\left[\frac{\sin\left(\frac{tg\,x-\sin x}{2}\right)}{\frac{tg\,x-\sin x}{2}}\cdot\frac{tg\,x-\sin x}{2x^3}\right]=\lim_{x\to 0}\frac{\sin\left(\frac{tg\,x-\sin x}{2}\right)}{\frac{tg\,x-\sin x}{2}}\cdot$$

$$\cdot\lim_{x\to 0}\frac{tg\,x-\sin x}{x^3}=1\cdot\lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^3\cos x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\lim_{x\to 0}\frac{1}{\cos x}=1\cdot\frac{1}{2}\cdot 1=\frac{1}{2}$$

Celkem tedy

$$\lim_{x\to 0}\frac{tg(tg\,x)-\sin(\sin x)}{tg\,x-\sin x}=2\cdot\frac{1}{2}+2\cdot\frac{1}{2}=\underline{2}.$$

1.2.85 $\lim\limits_{x\to b}\dfrac{a^x-a^b}{x-b},\quad a\rangle 0$

Řešení:

$$\lim_{x\to b}\frac{a^x-a^b}{x-b}=\lim_{x\to b}\frac{e^{x\ln a}-e^{b\ln a}}{x-b}=\lim_{x\to b}e^{b\ln a}\,\frac{e^{(x-b)\ln a}-1}{x-b}=$$

$$=e^{b\ln a}\cdot\lim_{x\to b}\ln a\frac{e^{(x-b)\ln a}-1}{(x-b)\ln a}=a^b\ln a\cdot\lim_{x\to b}\ln a\frac{e^{(x-b)\ln a}-1}{(x-b)\ln a}=a^b\ln a\cdot 1=\underline{a^b\ln a}.$$

1.2.86 $\lim\limits_{x\to 0}\dfrac{a^{x^2}-b^{x^2}}{(a^x-b^x)^2},\quad a\rangle 0,\ b\rangle 0$

Řešení:

$$\lim_{x\to 0}\frac{a^{x^2}-b^{x^2}}{(a^x-b^x)^2}=\lim_{x\to 0}\frac{\frac{a^{x^2}-b^{x^2}}{x^2}}{\left(\frac{a^x-b^x}{x}\right)^2}=\frac{\lim\limits_{x\to 0}\frac{a^{x^2}-b^{x^2}}{x^2}}{\left(\lim\limits_{x\to 0}\frac{a^x-b^x}{x}\right)^2}$$

$$\lim_{x\to 0}\frac{a^{x^2}-b^{x^2}}{x^2}=\lim_{x\to 0}\frac{\left(a^{x^2}-1\right)+\left(1-b^{x^2}\right)}{x^2}=\lim_{x\to 0}\frac{a^{x^2}-1}{x^2}-\lim_{x\to 0}\frac{b^{x^2}-1}{x^2}=\ln a-\ln b=\ln\frac{a}{b}$$

na základě výsledku příkladu 87.

$$\lim_{x\to 0}\frac{a^{x}-b^{x}}{x}=\lim_{x\to 0}\frac{\left(a^{x}-1\right)+\left(1-b^{x}\right)}{x}=\lim_{x\to 0}\frac{a^{x}-1}{x}-\lim_{x\to 0}\frac{b^{x}-1}{x}=\ln a-\ln b=\ln\frac{a}{b}$$

opět na základě výsledku příkladu 87. Obě předchozí limity je ovšem možno vypočíst i trochu jinak. Bez příkladu 87. se ovšem ani zde neobejdeme

$$\lim_{x\to 0}\frac{a^{x^2}-b^{x^2}}{x^2}=\lim_{x\to 0}\left[b^{x^2}\cdot\frac{\left(\frac{a}{b}\right)^{x^2}-1}{x^2}\right]=\lim_{x\to 0}b^{x^2}\cdot\lim_{x\to 0}\frac{\left(\frac{a}{b}\right)^{x^2}-1}{x^2}=1\cdot\ln\frac{a}{b}=\ln\frac{a}{b}$$

$$\lim_{x\to 0}\frac{a^{x}-b^{x}}{x}=\lim_{x\to 0}\left[b^{x}\cdot\frac{\left(\frac{a}{b}\right)^{x}-1}{x}\right]=\lim_{x\to 0}b^{x}\cdot\lim_{x\to 0}\frac{\left(\frac{a}{b}\right)^{x}-1}{x}=1\cdot\ln\frac{a}{b}=\ln\frac{a}{b}$$

Vychází tedy

$$\lim_{x\to 0}\frac{a^{x^2}-b^{x^2}}{\left(a^{x}-b^{x}\right)^2}=\frac{\ln\frac{a}{b}}{\left(\ln\frac{a}{b}\right)^2}=\frac{1}{\ln\frac{a}{b}}.$$

1.2.87 $\lim_{x\to a}\frac{\ln x-\ln a}{x-a}$

Řešení:

$$\lim_{x\to a}\frac{\ln x-\ln a}{x-a}=\lim_{x\to a}\frac{\ln\frac{x}{a}}{a\left(\frac{x}{a}-1\right)}=\frac{1}{a}\lim_{x\to a}\frac{\ln\left[1+\left(\frac{x}{a}-1\right)\right]}{\frac{x}{a}-1}$$

Nyní použijeme větu o limitě složené funkce. Vnitřní funkce je $g(x)=\frac{x}{a}-1$ a vnější funkci $f(y)=\frac{\ln(1+y)}{y}$.

$\lim_{x\to a}\left(\frac{x}{a}-1\right)=0$, $\lim_{y\to 0}\frac{\ln(1+y)}{y}=1$ Pro složenou funkci

$f(g(x))=\frac{\ln\left[1+\left(\frac{x}{a}-1\right)\right]}{\frac{x}{a}-1}$ tedy platí

$$\lim_{x \to a} = \frac{\ln\left[1+\left(\frac{x}{a}-1\right)\right]}{\frac{x}{a}-1} = \lim_{y \to 0} \frac{\ln(1+y)}{y} = 1$$

Celkem tedy vychází

$$\lim_{x \to a} \frac{\ln x - \ln a}{x-a} = \frac{1}{a} \cdot 1 = \underline{\frac{1}{a}}.$$

1.2.88 $\lim_{x \to a} \frac{x^b - a^b}{x-a}, a \rangle 0$

__Řešení:__

$$\lim_{x \to a} \frac{x^b - a^b}{x-a} = \lim_{x \to a} \frac{e^{b\ln x} - e^{b\ln a}}{x-a} = \lim_{x \to a} e^{b\ln a} \frac{e^{b(\ln x - \ln a)} - 1}{x-a} =$$

$$= e^{b\ln a} \cdot \lim_{x \to a} \left[\frac{e^{b(\ln x - \ln a)} - 1}{b(\ln x - \ln a)} \cdot \frac{b(\ln x - \ln a)}{x-a}\right] =$$

$$= a^b \cdot b \cdot \lim_{x \to a} \frac{e^{b(\ln x - \ln a)} - 1}{b(\ln x - \ln a)} \cdot \lim_{x \to a} \frac{\ln x - \ln a}{x-a} = a^b \cdot b \cdot 1 \cdot \frac{1}{a} = \underline{b \cdot a^{b-1}}.$$