3.1 Výpočet neurčitého integrálu

3.1.88

$$\int \ln x dx = \begin{bmatrix} u = \ln x & v' = 1 \\ u' = \frac{1}{x} & v = x \end{bmatrix} = x\ln x - \int \frac{x}{x} dx = x \ln x - x + C = \underline{x\left(\ln x - 1\right) + C}$$

3.1.89

$$\int \left(\frac{\ln x}{x}\right)^2 dx = \begin{bmatrix} u = \ln^2 x & v' = \frac{1}{x^2} \\ u' = \frac{2\ln x}{x} & v = -\frac{1}{x} \end{bmatrix} = -\frac{1}{x}\ln^2 x + 2\int \frac{\ln x}{x^2} dx = \begin{bmatrix} u = \ln x & v' = \frac{1}{x^2} \\ u' = \frac{1}{x} & v = -\frac{1}{x} \end{bmatrix} =$$

$$= -\frac{1}{x}\ln^2 x - \frac{2}{x}\ln x + 2\int \frac{1}{x^2} dx = -\frac{1}{x}\ln^2 x - \frac{2}{x}\ln x - 2\frac{1}{x} + C = \underline{-\frac{1}{x}\left(\ln^2 x + 2\ln x + 2\right) + C}$$

3.1.90

$$\int \ln\left(x + \sqrt{1+x^2}\right) dx = \begin{bmatrix} u = \ln\left(x + \sqrt{1+x^2}\right) & v' = 1 \\ u' = \frac{1}{\sqrt{1+x^2}} & v = x \end{bmatrix} = x\ln\left(x + \sqrt{1+x^2}\right) -$$

$$- \int \frac{x}{\sqrt{1+x^2}} dx = \underline{x\ln\left(x + \sqrt{1+x^2}\right) - \sqrt{1+x^2} + C}$$

3.1.91

$$\int \frac{x\ln\left(x + \sqrt{1+x^2}\right)}{\sqrt{1+x^2}} dx = \begin{bmatrix} u = \ln\left(x + \sqrt{1+x^2}\right) & v' = \frac{x}{\sqrt{1+x^2}} \\ u' = \frac{1}{\sqrt{1+x^2}} & v = \sqrt{1+x^2} \end{bmatrix} = \sqrt{1+x^2}\ln\left(x + \sqrt{1+x^2}\right) -$$

$$- \int dx = \underline{\sqrt{1+x^2}\ln\left(x + \sqrt{1+x^2}\right) - x + C}$$

3.1.92

$$\int \ln^2\left(x + \sqrt{1+x^2}\right) dx = \begin{bmatrix} u = \ln^2\left(x + \sqrt{1+x^2}\right) & v' = 1 \\ u' = \frac{2\ln\left(x + \sqrt{1+x^2}\right)}{\sqrt{1+x^2}} & v = x \end{bmatrix} = x\ln^2\left(x + \sqrt{1+x^2}\right) -$$

$$- 2\int \frac{x\ln\left(x + \sqrt{1+x^2}\right)}{\sqrt{1+x^2}} dx = \underline{x\ln^2\left(x + \sqrt{1+x^2}\right) - 2\sqrt{1+x^2}\ln\left(x + \sqrt{1+x^2}\right) + 2x + C}$$

3.1.93

$\int \ln\left(\sqrt{1-x}+\sqrt{1+x}\right)dx$

$Df=\langle -1,1\rangle$, *pro* $x\neq 0$ *použijeme metodu per partes*: $v'=1,\ v=x,\ u=\ln\left(\sqrt{1-x}+\sqrt{1+x}\right),$

$$u'=\frac{1}{2}\frac{1}{\sqrt{1-x}+\sqrt{1+x}}\frac{\sqrt{1-x}-\sqrt{1+x}}{\sqrt{1+x}\sqrt{1-x}}=\frac{1}{2}\frac{\left(\sqrt{1-x}+\sqrt{1+x}\right)^2}{-2x\sqrt{1-x^2}}=-\frac{1}{2}\frac{1-\sqrt{1-x^2}}{x\sqrt{1-x^2}}$$

$$F(x)=\int \ln\left(\sqrt{1-x}+\sqrt{1+x}\right)dx=x\ln\left(\sqrt{1-x}+\sqrt{1+x}\right)+\frac{1}{2}\int\frac{1-\sqrt{1-x^2}}{\sqrt{1-x^2}}dx=$$

$$=\underline{x\ln\left(\sqrt{1-x}+\sqrt{1+x}\right)+\frac{1}{2}\arcsin x-\frac{1}{2}x+C}$$

Pr*otože* $F(0)$ *existuje a* $\lim_{x\to 0}F(x)=F(0)$, *je* $F(x)$ *spojitá na* $(-1,1)$

3.1.94

$$\int \sin x\cdot\ln(tg\,x)dx=\begin{bmatrix} u=\ln(tg\,x) & v'=\sin x\\ u'=\dfrac{1}{tg\,x\cdot\cos^2 x} & v=-\cos x\end{bmatrix}=-\cos x\cdot\ln(tg\,x)+\int\frac{1}{\sin x}dx=$$

$$=\underline{-\cos x\cdot\ln(tg\,x)+\ln\left|tg\frac{x}{2}\right|+C}$$

3.1.95

$$\int xe^{-x}dx=\begin{bmatrix} u=x & v'=e^{-x}\\ u'=1 & v=-e^{-x}\end{bmatrix}=-xe^{-x}+\int e^{-x}dx=-xe^{-x}-e^{-x}+C=$$

$$=\underline{-(x+1)e^{-x}+C}$$

3.1.96

$$\int x^3e^{-x^2}dx=\begin{bmatrix} u=x^2 & v'=x\,e^{-x^2}\\ u'=2x & v=-\dfrac{1}{2}e^{-x^2}\end{bmatrix}=-\frac{x^2}{2}e^{-x^2}+\int x\,e^{-x^2}dx=-\frac{x^2}{2}e^{-x^2}-\frac{1}{2}e^{-x^2}+C=$$

$$=\underline{-\frac{1}{2}\left(1+x^2\right)e^{-x^2}+C}$$

3.1.97

$\int e^{\sqrt{x}}dx$

$t=\varphi(x)=\sqrt{x}$ $\qquad I=(0,\infty)\xrightarrow{\varphi}J=(0,\infty)$

$\varphi'(x)=\dfrac{1}{2\sqrt{x}}$ $\qquad y=R$

$$\int e^{\sqrt{x}}dx=2\int\frac{\sqrt{x}}{2\sqrt{x}}e^{\sqrt{x}}dx$$

$$F(t)=2\int t\,e^t\,dt=\begin{bmatrix} u=t & v'=e^t \\ u'=1 & v=e^t \end{bmatrix}=2t\,e^t-2\int e^t\,dt=2t\,e^t-2e^t+C$$

$$F(\varphi(x))=\underline{2\left(\sqrt{x}-1\right)e^{\sqrt{x}}+C}$$

3.1.98

$$\int x^2\,e^{\sqrt{x}}\,dx$$

$$t=\varphi(x)=\sqrt{x} \qquad\qquad I=(0,\infty)\xrightarrow{\varphi}J=(0,\infty)$$

$$\varphi'(x)=\frac{1}{2\sqrt{x}} \qquad\qquad y=R$$

$$\int x^2\,e^{\sqrt{x}}\,dx=2\int\frac{x^2\sqrt{x}}{2\sqrt{x}}e^{\sqrt{x}}\,dx,\quad z\ rovnice\ substituce:\ x^2=t^4$$

$$F(t)=2\int t^5\,e^t\,dt=\begin{bmatrix} u=t^5 & v'=e^t \\ u'=5t^4 & v=e^t \end{bmatrix}=2t^5\,e^t-10\int t^4e^t\,dt=\begin{bmatrix} u=t^4 & v'=e^t \\ u'=4t^3 & v=e^t \end{bmatrix}=$$

$$=2t^5e^t-10t^4e^t+40\int t^3e^t\,dt=\begin{bmatrix} u=t^3 & v'=e^t \\ u'=3t^2 & v=e^t \end{bmatrix}=2t^5e^t-10t^4e^t+40t^3e^t-$$

$$-120\int t^2e^t\,dt=\begin{bmatrix} u=t^2 & v'=e^t \\ u'=2t & v=e^t \end{bmatrix}=2t^5e^t-10t^4e^t+40t^3e^t-120t^2e^t+240\int t\,e^t\,dt=$$

$$=\begin{bmatrix} u=t & v'=e^t \\ u'=1 & v=e^t \end{bmatrix}=2t^5e^t-10t^4e^t+40t^3e^t-120t^2e^t+240t\,e^t-240\int e^t\,dt=$$

$$=2e^t\left(t^5-5t^4-60t^2+120t-120\right)+C$$

$$F(\varphi(x))=\underline{2e^{\sqrt{x}}\left(\sqrt{x}^5-5x^2+20\sqrt{x}^3-60x+120\sqrt{x}-120\right)+C}$$

3.1.99

$$\int x\cos x\,dx=\begin{bmatrix} u=x & v'=\cos x \\ u'=1 & v=\sin x \end{bmatrix}=x\sin x-\int\sin x\,dx=\underline{x\sin x+\cos x+C}$$

3.1.100

$$\int x^2\sin 2x\,dx=\begin{bmatrix} u=x^2 & v'=\sin 2x \\ u'=2x & v=-\frac{1}{2}\cos 2x \end{bmatrix}=-\frac{x^2}{2}\cos 2x+\int x\cos 2x\,dx=$$

$$=\begin{bmatrix} u=x & v'=\cos 2x \\ u'=1 & v=\frac{1}{2}\sin 2x \end{bmatrix}=-\frac{x^2}{2}\cos 2x+\frac{x}{2}\sin 2x-\frac{1}{2}\int\sin 2x\,dx=$$

$$=\underline{-\frac{x^2}{2}\cos 2x+\frac{x}{2}\sin 2x+\frac{1}{4}\cos 2x+C}$$

3.1.101

$$\int x\sin^2 x\,dx = \begin{bmatrix} u = x & v' = \sin^2 x \\ u' = 1 & v = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) \end{bmatrix} = \frac{1}{2}x\left(x - \frac{\sin 2x}{2}\right) - \frac{1}{2}\int\left(x - \frac{\sin 2x}{2}\right)dx =$$

$$= \frac{1}{2}x^2 - \frac{x\sin 2x}{4} - \frac{1}{4}x^2 - \frac{\cos 2x}{8} + C = \underline{\frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C}$$

3.1.102

$$\int x\sin\sqrt{x}\,dx$$

$$t = \varphi(x) = \sqrt{x} \qquad I = (0,\infty) \xrightarrow{\varphi} J = (0,\infty)$$

$$\varphi'(x) = \frac{1}{2\sqrt{x}} \qquad y = R$$

$$\int x\sin\sqrt{x}\,dx = 2\int \frac{x\sqrt{x}}{2\sqrt{x}}\sin\sqrt{x}\,dx, \quad z\ rovnice\ substituce: x = t^2$$

$$F(t) = 2\int t^3\sin t\ dt = \begin{bmatrix} u = t^3 & v' = \sin t \\ u' = 3t^2 & v = -\cos t \end{bmatrix} = 2t^3(-\cos t) + 6\int t^2\cos t\,dt =$$

$$= \begin{bmatrix} u = t^2 & v' = \cos t \\ u' = 2t & v = \sin t \end{bmatrix} = -2t^3\cos t + 6t^2\sin t - 12\int t\sin t\,dt = \begin{bmatrix} u = t & v' = \sin t \\ u' = 1 & v = -\cos t \end{bmatrix} =$$

$$= -2t^3\cos t + 6t^2\sin t + 12t\cos t - 12\sin t + C$$

$$F(\varphi(x)) = \underline{2\sqrt{x}(6 - x)\cos\sqrt{x} + 6(x - 2)\sin\sqrt{x} + C}$$

3.1.103

$$\int \cos^2\sqrt{x}\,dx$$

$$t = \varphi(x) = \sqrt{x} \qquad I = (0,\infty) \xrightarrow{\varphi} J = (0,\infty)$$

$$\varphi'(x) = \frac{1}{2\sqrt{x}} \qquad y = R$$

$$\int \cos^2\sqrt{x}\,dx = 2\int \frac{\sqrt{x}}{2\sqrt{x}}\cos^2\sqrt{x}\,dx$$

$$F(t) = 2\int t\cos^2 t\ dt = \begin{bmatrix} u = t & v' = \cos^2 t \\ u' = 1 & v = \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right) \end{bmatrix} = t^2 + \frac{t\sin 2t}{2} - \int\left(t + \frac{\sin 2t}{2}\right)dt =$$

$$= \frac{1}{2}t^2 + \frac{t}{2}\sin 2t + \frac{1}{4}\cos 2t + C$$

$$F(\varphi(x)) = \underline{\frac{x}{2} + \frac{\sqrt{x}}{2}\sin 2\sqrt{x} + \frac{1}{4}\cos 2\sqrt{x} + C}$$

3.1.104

$$\int arctg\,x\,dx = \begin{bmatrix} u = arctg\,x & v' = 1 \\ u' = \dfrac{1}{1+x^2} & v = x \end{bmatrix} = x\,arctg\,x - \int \frac{x}{1+x^2}\,dx = \underline{x\,arctg\,x - \frac{1}{2}\ln\left(1+x^2\right) + C}$$

3.1.105

$$\int arctg\sqrt{x}\,dx$$

$t = \varphi(x) = \sqrt{x}$ $\qquad I = (0,\infty) \xrightarrow{\varphi} J = (0,\infty)$

$\varphi'(x) = \dfrac{1}{2\sqrt{x}}$ $\qquad y = R$

$$\int \ddot{a}rctg\sqrt{x}\,dx = 2\int \frac{\sqrt{x}}{2\sqrt{x}}\,arctg\sqrt{x}\,dx$$

$$F(t) = 2\int t\,arctg\,t\,dt = \begin{bmatrix} u = arctg\,t & v' = t \\ u' = \dfrac{1}{1+t^2} & v = \dfrac{t^2}{2} \end{bmatrix} = t^2\,arctg\,t - \int \frac{t^2}{1+t^2}\,dt =$$

$$= t^2\,arctg\,t - t + arctg\,t + C = \left(t^2+1\right)arctg\,t - t + C$$

$$F(\varphi(x)) = \underline{(x+1)\,arctg\sqrt{x} - \sqrt{x} + C}$$

3.1.106

$$\int x\,arctg^2 x\,dx = \begin{bmatrix} u = arctg^2\,x & v' = x \\ u' = 2(arctg\,x)\dfrac{1}{1+x^2} & v = \dfrac{x^2}{2} \end{bmatrix} = \frac{x^2}{2}arctg^2 x - \int \frac{x^2}{1+x^2}\,arctg\,x\,dx =$$

$$= \frac{x^2}{2}\,arctg^2 x - \int arctg\,x\,dx + \int \frac{arctg\,x}{1+x^2}\,dx = \underline{\frac{x^2+1}{2}arctg^2 x - x\,arctg\,x + \frac{1}{2}\ln\left(1+x^2\right) + C}$$

3.1.107

$$\int x\,arctg(x+1)dx = \begin{bmatrix} u = arctg(x+1) & v' = x \\ u' = \dfrac{1}{(x+1)^2+1} & v = \dfrac{x^2}{2} \end{bmatrix} = \frac{x^2}{2}arctg(x+1) - \frac{1}{2}\int \frac{x^2}{(x+1)^2+1}\,dx =$$

$$= \frac{x^2}{2}\,arctg(x+1) - \frac{1}{2}\int arctg\,x\,dx + \int \frac{x^2+2x+2-2x-2}{x^2+2x+2}\,dx =$$

$$= \underline{\frac{x^2}{2}arctg(x+1) - \frac{1}{2}x + \frac{1}{2}\ln\left(x^2+2x+2\right) + C}$$

Je možné řešit též substitucí $t = x+1$ a pak per partes.

3.1.108

$$\int \frac{x\ arctg\,x}{\sqrt{1+x^2}}\,dx = \begin{bmatrix} u = arctg\,x & v' = \dfrac{x}{\sqrt{1+x^2}} \\ u' = \dfrac{1}{1+x^2} & v = \sqrt{1+x^2} \end{bmatrix} = \sqrt{1+x^2}\ arctg\,x - \int \frac{1}{\sqrt{1+x^2}}\,dx =$$

$$= \underline{\sqrt{1+x^2}\ arctg\,x - \ln\left|x+\sqrt{1+x^2}\right| + C}$$

3.1.109

$$\int \arcsin x\ dx = \begin{bmatrix} u = \arcsin x & v' = 1 \\ u' = \dfrac{1}{\sqrt{1-x^2}} & v = x \end{bmatrix} = x\ \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx =$$

$$= \underline{x\ \arcsin x + \sqrt{1-x^2} + C}$$

3.1.110

$$\int x^2 \arccos x\ dx = \begin{bmatrix} u = \arccos x & v' = x^2 \\ u' = -\dfrac{1}{\sqrt{1-x^2}} & v = \dfrac{x^3}{3} \end{bmatrix} = \frac{x^3}{3}\ \arccos x + \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx = (\otimes)$$

$\int \frac{x^3}{\sqrt{1-x^2}}\,dx$ *řešíme substitucí*: $t = \varphi(x) = \sqrt{1-x^2}$ $\qquad I = (-1,1) \xrightarrow{\varphi} J = (0,1\rangle$

$\varphi'(x) = \frac{-x}{\sqrt{1-x^2}}$ $\qquad y = R$

z rovnice substituce: $x^2 = 1 - t^2$

$$F(t) = -\int (1-t^2)\,dt = \frac{t^3}{3} - t + C$$

$$F(\varphi(x)) = \frac{\left(\sqrt{1-x^2}\right)^3}{3} - \sqrt{1-x^2} + C = -\frac{2+x^2}{3}\sqrt{1-x^2} + C$$

Výsledek: $(\otimes) = \underline{\frac{x^3}{3}\arccos x - \frac{2+x^2}{9}\sqrt{1-x^2} + C}$

3.1.111

$$\int \frac{\arcsin x}{x^2}\,dx = \begin{bmatrix} u = \arcsin x & v' = \dfrac{1}{x^2} \\ u' = \dfrac{1}{\sqrt{1-x^2}} & v = -\dfrac{1}{x} \end{bmatrix} = -\frac{1}{x}\ \arcsin x + \int \frac{1}{x\sqrt{1-x^2}}\,dx = (\otimes)$$

$$\int \frac{1}{x\sqrt{1-x^2}}\,dx \quad \textit{řešíme substitucí}:\ t=\varphi(x)=\sqrt{1-x^2} \qquad I=(-1,0)\bigcup(0,1) \quad J=(0,1\rangle$$

$$\varphi'(x)=\frac{-x}{\sqrt{1-x^2}} \qquad y=(-\infty,-1)\bigcup(-1,1)\bigcup(1,\infty)$$

$$F(t)=-\int \frac{1}{1-t^2}\,dt=\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+C$$

$$F(\varphi(x))=\frac{1}{2}\ln\left|\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right|+C=\ln\left|\frac{\sqrt{1-x^2}-1}{x}\right|+C$$

$$\textit{Výsledek}:\ (\otimes)=\underline{-\frac{1}{x}\arcsin x+\ln\left|\frac{\sqrt{1-x^2}-1}{x}\right|+C}$$

3.1.112

$$\int(\arcsin x)^2\,dx=\begin{bmatrix} u=\arcsin^2 x & v'=1 \\ u'=\dfrac{2\arcsin x}{\sqrt{1-x^2}} & v=x \end{bmatrix}=x\arcsin^2 x-2\int\frac{x\arcsin x}{\sqrt{1-x^2}}\,dx=$$

$$\begin{bmatrix} u=\arcsin x & v'=\dfrac{x}{\sqrt{1-x^2}} \\ u'=\dfrac{1}{\sqrt{1-x^2}} & v=-\sqrt{1-x^2} \end{bmatrix}=\underline{x\arcsin^2 x+2\sqrt{1-x^2}\arcsin x-2x+C}$$

3.1.113

$$F(x)=\int \arcsin\frac{2\sqrt{x}}{x+1}\,dx$$

$$Df=\langle 0,\infty)$$

$F(x)$ *budeme řešit pomocí per partes, uvažujme však* Df' *funkce* $f(x)=\arcsin\dfrac{2\sqrt{x}}{x+1}$.

$$f'(x)=\frac{1-x}{\sqrt{(1-x)^2}}\,\frac{1}{\sqrt{x}(x+1)}=\frac{1}{\sqrt{x}(x+1)}\operatorname{sgn}(1-x),\ \ Df'=(0,1)\bigcup(1,\infty).$$

$\underline{x\in(0,1)}$

$$F_1(x)=\int \arcsin\frac{2\sqrt{x}}{x+1}\,dx=\left[\begin{array}{ll} u=\arcsin\frac{2\sqrt{x}}{x+1} & v'=1 \\ u'=\frac{1}{\sqrt{x}(x+1)} & v=x+1 \end{array}\right]=(x+1)\arcsin\frac{2\sqrt{x}}{x+1}-\int\frac{1}{\sqrt{x}}dx=$$

$$=(x+1)\arcsin\frac{2\sqrt{x}}{x+1}-2\sqrt{x}+C_1$$

(*Pozn. je lepší uvažovat* $v=x+1$ *než* $v=x$, *protože postup bude jednodušší*; *pro* $v=x$ *dos*tan*eme pochopite*ln*ě stejný výsledek*.)

$x\in(1,\infty)$

Stejným postupem jako pro $x\in(0,1)$

$$F_2(x)=(x+1)\arcsin\frac{2\sqrt{x}}{x+1}+2\sqrt{x}+C_2$$

Pr*otože* $\lim_{x\to1-}F_1(x)=\pi-2+C_1$ *a* $\lim_{x\to1+}F_2(x)=\pi+2+C_2$, *zvolíme kons*tan*ty* C_1 *a* C_2 *tak*, *aby* $F(x)$ *byla spojitá na* $(0,\infty)$:

$$F(x)=\left\langle\begin{array}{ll} (x+1)\arcsin\frac{2\sqrt{x}}{x+1}-2\sqrt{x}+4+C, & pro\ x\in(0,1\rangle \\ (x+1)\arcsin\frac{2\sqrt{x}}{x+1}+2\sqrt{x}+C, & pro\ x\in(1,\infty) \end{array}\right.$$

3.1.114

$$\int\sqrt{1-x^2}\,\arcsin x\,dx$$

$t=\varphi(x)=\arcsin x$ $\qquad I=(-1,1)\xrightarrow{\varphi}J=\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$\varphi'(x)=\frac{1}{\sqrt{1-x^2}}$ $\qquad y=R$

$$\int\sqrt{1-x^2}\,\arcsin x\,dx=\int\frac{1-x^2}{\sqrt{1-x^2}}\arcsin x\,dx,\quad z\ rovnice\ substituce: x=\sin t$$

$$F(t)=\int t\cos^2 t\,dt=\frac{t^2}{4}+\frac{t}{4}\sin 2t+\frac{1}{8}\cos 2t+C_1=\frac{t^2}{4}+\frac{t}{2}\sqrt{1-\sin^2 t}\,\sin t+\frac{1}{8}\left(1-2\sin^2 t\right)+C_1$$

$$F(\varphi(x))=\frac{1}{4}\arcsin^2 x+\frac{1}{2}x\sqrt{1-x^2}\,\arcsin x-\frac{1}{4}x^2+C,\ \ kde\ C=C_1+\frac{1}{8}$$

3.1.115

$$\int \frac{x^3 \arccos x}{\sqrt{1-x^2}}\,dx = \begin{bmatrix} u = \arccos x & v' = \frac{x^3}{\sqrt{1-x^2}} \\ u' = -\frac{1}{\sqrt{1-x^2}} & v = -\frac{2+x^2}{3}\sqrt{1-x^2} \end{bmatrix} = -\frac{2+x^2}{3}\arccos x -$$

$$-\int \frac{2+x^2}{3}\,dx = \underline{-\frac{2+x^2}{3}\arccos x - \frac{2}{3}x - \frac{1}{9}x^3 + C}$$

3.1.116

$$\int \frac{x^2}{\left(1+x^2\right)^2}\,dx$$

Volíme $u = x,\ v' = \frac{x}{\left(1+x^2\right)^2}$, *potom* $v = \int \frac{x}{\left(1+x^2\right)^2}\,dx$. *Pomocí substituce* $\varphi(x) = 1 + x^2$ *získáme* $v = -\frac{1}{2}\,\frac{1}{1+x^2}$. *Potom* $\int \frac{x^2}{\left(1+x^2\right)^2}\,dx = -\frac{x}{2}\,\frac{1}{1+x^2} + \frac{1}{2}\int \frac{1}{1+x^2}\,dx =$

$$\underline{= -\frac{x}{2}\,\frac{1}{1+x^2} + \frac{1}{2}\,arctg\,x + C}$$

3.1.117

$$\int \frac{dx}{\left(a^2+x^2\right)^2} \quad pro \ \ a \neq 0$$

$$\frac{1}{a^2}\int \frac{a^2}{\left(a^2+x^2\right)^2}\,dx = \frac{1}{a^2}\int \frac{a^2+x^2-x^2}{\left(a^2+x^2\right)^2}\,dx = \frac{1}{a^2}\int \frac{1}{a^2+x^2}\,dx - \frac{1}{a^2}\int \frac{x^2}{\left(a^2+x^2\right)^2}\,dx =$$

$$= \frac{1}{a^4}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}\,dx - \frac{1}{a^4}\int \frac{\left(\frac{x}{a}\right)^2}{\left(1+\left(\frac{x}{a}\right)^2\right)^2}\,dx = \frac{1}{a^3}\,arctg\,\frac{x}{a} - \left[-\frac{1}{2}\,\frac{x}{a}\,\frac{1}{1+\left(\frac{x}{a}\right)^2}\right] -$$

$$-\frac{1}{a^3}\,\frac{1}{2}\,arctg\,\frac{x}{a} + C = \underline{\frac{1}{2a^3}\,arctg\,\frac{x}{a} + \frac{x}{2a^2\left(a^2+x^2\right)} + C}$$

3.1.118

$$\int \sqrt{a^2-x^2}\,dx = \begin{bmatrix} u = \sqrt{a^2-x^2} & v' = 1 \\ u' = \frac{-x}{\sqrt{a^2-x^2}} & v = x \end{bmatrix} = x\sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}\,dx =$$

$$= x\sqrt{a^2 - x^2} + \int \frac{x^2 - a^2 + a^2}{\sqrt{a^2 - x^2}}dx = x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\,dx + \frac{a^2}{|a|}\int \frac{dx}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} =$$

$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\,dx + a^2 (\operatorname{sgn} a)\arcsin\frac{x}{a},\ \textit{kde}\ (\operatorname{sgn} a)\arcsin\frac{x}{a} = \arcsin\frac{x}{|a|}.$$

Nyní postupujeme rovnicí pro neznámou $\int \sqrt{a^2 - x^2}\,dx$ *a dosáváme* $\int \sqrt{a^2 - x^2}\,dx =$

$$= \underline{\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{|a|} + C}$$

3.1.119

$\int \sqrt{a^2 + x^2}\,dx$, *kde* $(a \neq 0)$

$$\int \sqrt{a^2 + x^2}\,dx = \begin{bmatrix} u = \sqrt{a^2 + x^2} & v' = 1 \\ u' = \dfrac{x}{\sqrt{a^2 + x^2}} & v = x \end{bmatrix} = x\sqrt{a^2 + x^2} - \int \frac{x^2}{\sqrt{a^2 + x^2}}dx =$$

$$= x\sqrt{a^2 + x^2} - \int \frac{x^2 + a^2 - a^2}{\sqrt{a^2 + x^2}}dx = x\sqrt{a^2 + x^2} - \int \sqrt{a^2 + x^2} + \frac{a^2}{|a|}\int \frac{dx}{\sqrt{1 + \left(\frac{x}{a}\right)^2}} =$$

$$= x\sqrt{a^2 + x^2} - \int \sqrt{a^2 + x^2}\,dx + a^2 \ln\left(x + \sqrt{a^2 + x^2}\right)$$

Platí totiž: $\displaystyle\int \frac{dx}{\sqrt{a^2 + x^2}} = \frac{1}{|a|}\int \frac{dx}{\sqrt{1 + \left(\frac{x}{a}\right)^2}} = \frac{a}{|a|}\ln\left|\frac{x}{a} + \frac{\sqrt{a^2 + x^2}}{|a|}\right| = \otimes$

pro $a \rangle 0$

$\otimes = \ln\left(x + \sqrt{a^2 + x^2}\right) - \ln a$, *neboť* $x + \sqrt{a^2 + x^2} \rangle 0$ *pro* $x \in R$

pro $a \langle 0$

$$\otimes = -\ln\left|x - \sqrt{a^2 + x^2}\right| + \ln|a| = \ln\frac{1}{\left|x - \sqrt{a^2 + x^2}\right|} + \ln|a| = \ln\left|\frac{x + \sqrt{a^2 + x^2}}{-a^2}\right| + \ln|a| =$$

$$= \ln\left(x + \sqrt{a^2 + x^2}\right) - \ln|a|$$

Nyní postupujeme rovnicí pro neznámou $\int \sqrt{a^2 + x^2}\,dx$ *a dosáváme*:

$$\int \sqrt{a^2 + x^2}\,dx = \underline{= \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\ln\left(x + \sqrt{a^2 + x^2}\right) + C}$$

3.1.120

$\int x^2 \sqrt{a^2 + x^2}\, dx$, *kde* $(a \neq 0)$

$$\int x^2 \sqrt{a^2 + x^2}\, dx = \begin{bmatrix} u = \sqrt{a^2 + x^2} & v' = x^2 \\ u' = \dfrac{x}{\sqrt{a^2 + x^2}} & v = \dfrac{x^3}{3} \end{bmatrix} = \frac{x^3}{3}\sqrt{a^2 + x^2} - \frac{1}{3}\int \frac{x^4}{\sqrt{a^2 + x^2}}\, dx =$$

$$= \frac{x^3}{3}\sqrt{a^2 + x^2} - \frac{1}{3}\int x^2 \sqrt{a^2 + x^2}\, dx + \frac{a^2}{3}\int \sqrt{a^2 + x^2}\, dx - \frac{a^4}{3}\int \frac{1}{\sqrt{a^2 + x^2}}\, dx,\ \textit{kde jsme}$$

výraz $\dfrac{x^4}{\sqrt{a^2 + x^2}}$ *upravili* $\dfrac{x^4}{\sqrt{a^2 + x^2}} = \sqrt{a^2 + x^2}\ \dfrac{x^4}{a^2 + x^2} = \sqrt{a^2 + x^2}\left(x^2 - a^2 + \dfrac{a^4}{a^2 + x^2}\right)$

Nyní postupujeme rovnicí pro neznámou $\int x^2 \sqrt{a^2 + x^2}\, dx$ *a dosáváme*:

$$\int x^2 \sqrt{a^2 + x^2}\, dx = \underline{\frac{x}{8}\left(2x^2 + a^2\right)\sqrt{a^2 + x^2} - \frac{a^2}{8}\ln\left(x + \sqrt{a^2 + x^2}\right) + C}$$

3.1.121

$$\int \sin(\ln x)dx = \begin{bmatrix} u = \sin(\ln x) & v' = 1 \\ u' = \dfrac{\cos(\ln x)}{x} & v = x \end{bmatrix} = x\sin(\ln x) - \int \cos(\ln x)dx =$$

$$= \begin{bmatrix} u = \cos(\ln x) & v' = 1 \\ u' = \dfrac{-\sin(\ln x)}{x} & v = x \end{bmatrix} = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)dx$$

Nyní postupujeme rovnicí pro neznámou $\int \sin(\ln x)dx$ *a dosáváme*:

$$\int \sin(\ln x)dx = \underline{\frac{x}{2}\left[\sin(\ln x) - \cos(\ln x)\right] + C}$$

3.1.122

$$\int e^{ax}\cos bx\, dx = \begin{bmatrix} u = \cos bx & v' = e^{ax} \\ u' = -b\sin bx & v = \dfrac{1}{a}e^{ax} \end{bmatrix} = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\, dx =$$

$$= \begin{bmatrix} u = \sin bx & v' = e^{ax} \\ u' = b\cos bx & v = \dfrac{1}{a}e^{ax} \end{bmatrix} = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\frac{1}{a}e^{ax}\sin bx - \frac{b^2}{a^2}\int e^{ax}\cos bx\, dx$$

Nyní postupujeme rovnicí pro neznámou $\int e^{ax}\cos bx\, dx$ *a dosáváme*:

$$\int e^{ax}\cos bx\, dx = \underline{\frac{e^{ax}}{a^2 + b^2}\left(a\cos bx + b\sin bx\right) + C}$$

Analogicky: $\int e^{ax}\sin bx\, dx = \underline{\dfrac{e^{ax}}{a^2 + b^2}\left(a\sin bx - b\cos bx\right) + C}$

3.1.123

$$\int x\,e^{x}\sin x\,dx=\begin{bmatrix} u=x & v'=e^{x}\sin x \\ u'=1 & v=\frac{1}{2}e^{x}(\sin x-\cos x)\end{bmatrix}=\frac{1}{2}x\,e^{x}(\sin x-\cos x)-\frac{1}{2}\int e^{x}\sin x\,dx+$$

$$+\frac{1}{2}\int e^{x}\cos x\,dx=\frac{1}{2}x\,e^{x}(\sin x-\cos x)-\frac{1}{4}e^{x}(\sin x-\cos x)+\frac{1}{4}e^{x}(\cos x+\sin x)+C=$$

$$=\underline{\frac{1}{2}e^{x}(x\sin x-x\cos x+\cos x)+C}$$

3.1.124

$$\int x\,e^{x}\sin^{2}x\,dx=\int x\,e^{x}\frac{1-\cos 2x}{2}\,dx=\frac{1}{2}\int x\,e^{x}\,dx-\int x\,e^{x}\cos 2x\,dx$$

$$\int x\,e^{x}\,dx=x\,e^{x}-e^{x}+C_1$$

$$\int x\,e^{x}\cos 2x\,dx=\begin{bmatrix} u=x & v'=e^{x}\cos 2x \\ u'=1 & v=\frac{e^{x}}{5}(\cos 2x+2\sin 2x)\end{bmatrix}=\frac{x\,e^{x}}{5}(\cos 2x+2\sin 2x)-$$

$$-\frac{1}{5}\int e^{x}\cos 2x\,dx-\frac{2}{5}\int e^{x}\sin 2x\,dx=\frac{x\,e^{x}}{5}(\cos 2x+2\sin 2x)-\frac{e^{x}}{25}(\cos 2x+2\sin 2x)-$$

$$-\frac{2\,e^{x}}{25}(\sin 2x-2\cos 2x)+C_2=e^{x}\left[\frac{x}{5}(\cos 2x+2\sin 2x)+\frac{3}{25}\cos 2x-\frac{4}{25}\sin 2x\right]+C_2$$

$$\textit{Výsledek}:\ F(x)=\underline{\frac{e^{x}}{2}\left[x-1-\frac{x}{5}(\cos 2x+2\sin 2x)+\frac{1}{25}(4\sin 2x-3\cos 2x)\right]+C}$$

3.1.125

$$\int\frac{\ln(\sin x)}{\sin^{2}x}\,dx=\begin{bmatrix} u=\ln(\sin x) & v'=\frac{1}{\sin^{2}x} \\ u'=\frac{\cos x}{\sin x} & v=-\cot g\,x\end{bmatrix}=-(\cot g\,x)\ln(\sin x)+\int\frac{\cos^{2}x}{\sin^{2}x}\,dx=$$

$$=-(\cot g\,x)\ln(\sin x)+\int\frac{1-\sin^{2}x}{\sin^{2}x}\,dx=-(\cot g\,x)\ln(\sin x)-\cot g\,x-x+C=$$

$$=\underline{-[x+(\cot g\,x)\ln(e\sin x)]+C}$$

3.1.126

$$\int\frac{x}{\cos^{2}x}\,dx=\begin{bmatrix} u=x & v'=\frac{1}{\cos^{2}x} \\ u'=1 & v=tg\,x\end{bmatrix}=x\,tg\,x-\int tg\,x\,dx=\underline{x\,tg\,x+\ln|\cos x|+C}$$

3.1.127

$$\int \frac{x e^x}{(x+1)^2}\,dx = \begin{bmatrix} u = x e^x & v' = \dfrac{1}{(x+1)^2} \\ u' = x e^x + e^x & v = -\dfrac{1}{1+x} \end{bmatrix} = -\frac{x e^x}{1+x} + \int \frac{x e^x + e^x}{1+x}\,dx =$$

$$= -\frac{x e^x}{1+x} + e^x + C = \underline{\frac{e^x}{1+x} + C}$$

3.1.128

$$\int \left(1 - \frac{2}{x}\right)^2 e^x\,dx = \int e^x\,dx - \int \frac{4}{x} e^x\,dx + \int \frac{4}{x^2} e^x\,dx$$

Druhý integrál řešíme per partes :

$$\int \frac{e^x}{x}\,dx = \begin{bmatrix} u = \dfrac{1}{x} & v' = e^x \\ u' = -\dfrac{1}{x^2} & v = e^x \end{bmatrix} = \frac{e^x}{x} + \int \frac{e^x}{x^2}\,dx,$$ dále dosadíme do původního integrálu

a dostaneme : $\int \left(1 - \frac{2}{x}\right)^2 e^x\,dx = \underline{e^x - 4\frac{e^x}{x} + C}$

K výsledku bychom došli též řešením třetího integrálu per partes.

3.1.129

$$\int \ln\left[(x+a)^{x+a}(x+b)^{x+b}\right] \frac{dx}{(x+a)(x+b)} = \int \frac{\ln(x+a)}{x+b}\,dx + \int \frac{\ln(x+b)}{x+a}\,dx$$

První (nebo druhý) integrál řešíme per partes :

$$\int \frac{\ln(x+a)}{x+b}\,dx = \begin{bmatrix} u = \ln(x+a) & v' = \dfrac{1}{x+b} \\ u' = \dfrac{1}{x+a} & v = \ln(x+b) \end{bmatrix} = \ln(x+a)\cdot\ln(x+b) - \int \frac{\ln(x+b)}{x+a}\,dx$$

Dosadíme do původního integrálu a dostaneme :

$$\int \ln\left[(x+a)^{x+a}(x+b)^{x+b}\right] \frac{dx}{(x+a)(x+b)} = \underline{\ln(x+a)\cdot\ln(x+b) + C}$$

3.1.130

$$\int \sin^n x\,dx\ (n \ge 2) = \begin{bmatrix} u = \sin^{n-1} x & v' = \sin x \\ u' = (n-1)\sin^{n-2} x \cos x & v = -\cos x \end{bmatrix} = -\cos x \sin^{n-1} x +$$

$$+ (n-1)\int \sin^{n-2} x \cos^2 x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$$

Nyní řešíme rovnici pro neznámou $\int \sin^n x\,dx$ a dostaneme rekurentní vztah

$$\underline{\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx}$$

Analogicky dojdeme k rekurentnímu vzorci pro $\int \cos^n x\,dx$, $n \ge 2$:

$$\int \cos^n x\,dx = \underline{\frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx}$$

3.1.131

$$\int \frac{1}{\sin^n x}\,dx\ (n \geq 2) = \begin{bmatrix} u = \dfrac{1}{\sin^{n-2} x} & v' = \dfrac{1}{\sin^2 x} \\ u' = \dfrac{-(n-2)\cos x}{\sin^{n-1} x} & v = -\dfrac{\cos x}{\sin x} \end{bmatrix} = -\frac{\cos x}{\sin^{n-1} x} -$$

$$-(n-2)\int \frac{\cos^2 x}{\sin^n x}\,dx = -\frac{\cos x}{\sin^{n-1} x} - (n-2)\int \frac{1}{\sin^n x}\,dx + (n-2)\int \frac{1}{\sin^{n-2} x}\,dx$$

Nyní řešíme rovnici pro neznámou $\int \frac{1}{\sin^n x}\,dx$ a dostaneme rekurentní vzorec

$$\int \frac{1}{\sin^n x}\,dx = \underline{-\frac{1}{n-1}\,\frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{1}{\sin^{n-2} x}\,dx}$$

Analogicky dojdeme k rekurentnímu vzorci pro $\int \frac{1}{\cos^n x}\,dx$, $n \geq 2$:

$$\int \frac{1}{\cos^n x}\,dx = \underline{\frac{1}{n-1}\,\frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{1}{\cos^{n-2} x}\,dx}$$