3.1 Výpočet neurčitého integrálu

3.1.132 $\int \sqrt{\frac{e^x - 1}{e^x + 1}} dx$ $D_f = \langle 0, \infty)$

položíme $t = \sqrt{\frac{e^x - 1}{e^x + 1}}$ a volíme substituci:

$x = \psi(t) = \ln \frac{t^2 + 1}{1 - t^2}$ $J = (0,1) \xrightarrow{\varphi} I = (0, \infty)$

$\psi'(t) = \frac{4t}{(t^2 + 1)(1 - t^2)}$ $\psi^{-1}(x) = \sqrt{\frac{e^x - 1}{e^x + 1}}$

$$G(t) = \int t \frac{4t}{(t^2 + 1)(1 - t^2)} dt = 4 \int \frac{t^2}{(t^2 + 1)(1 - t^2)} dt$$

nalezneme rozklad na parciální zlomky: $\frac{t^2}{(t^2 + 1)(1 - t^2)} = \frac{At + b}{t^2 + 1} + \frac{C}{1 - t} + \frac{D}{1 + t}$

t = -1 : $D = \frac{1}{4}$

t = 1 : $C = \frac{1}{4}$

t^3 : $-A + C - D = 0 \Rightarrow A = 0$

t = 0 : $B + C + D = 0 \Rightarrow B = -\frac{1}{2}$

$$G(t) = \int \frac{1}{1 - t} dt + \int \frac{1}{1 + t} dt - 2 \int \frac{1}{1 + t^2} dt = \ln \frac{1 + t}{1 - t} - 2\ arctg\ t + C_0$$

$$G(\psi^{-1}(x)) = \ln \frac{\sqrt{e^x - 1} + \sqrt{e^x + 1}}{\sqrt{e^x + 1} - \sqrt{e^x - 1}} - 2\ arctg\ \sqrt{\frac{e^x - 1}{e^x + 1}} + C_0$$

$$\underline{G(\psi^{-1}(x)) = \ln\left(e^x + \sqrt{e^{2x} - 1}\right) - 2\ arctg \sqrt{\frac{e^x - 1}{e^x + 1}} + C}$$

3.1.133 $\int \sqrt{e^{2x} + 4e^x - 1}\, dx$ $D_f = \langle \ln\left(-2 + 2\sqrt{5}\right), \infty)$

volíme substituci: $t = \varphi(x) = e^x$ $I = \langle \ln\left(-2 + 2\sqrt{5}\right), \infty) \xrightarrow{\varphi} J = (-2 + 2\sqrt{5}, \infty)$

$\varphi'(x) = e^x$ $J = (-\infty, -2 - 2\sqrt{5}) \cup (-2 + 2\sqrt{5}, \infty)$

$$F(t) = \int \frac{\sqrt{t^2 + 4t - 1}}{t} dt = \int \frac{t + 4}{\sqrt{t^2 + 4t - 1}} dt - \int \frac{t}{t\sqrt{t^2 + 4t - 1}} dt =$$

$$= \frac{1}{2}\int \frac{2t+4}{\sqrt{t^2+4t-1}}\,dt + 2\int \frac{1}{\sqrt{t^2+4t-1}}\,dt - \int \frac{1}{t\sqrt{t^2+4t-1}}\,dt$$

$$F_1(t) = \frac{1}{2}\int \frac{2t+4}{\sqrt{t^2+4t-1}}dt = \frac{1}{2}\sqrt{t^2+4t-1} + C_1$$

$$F_2(t) = 2\int \frac{1}{\sqrt{t^2+4t-1}}\,dt = \frac{2}{\sqrt{5}}\int \frac{1}{\sqrt{\left(\frac{t+2}{\sqrt{5}}\right)^2+1}}\,dt = 2\ln\left(\frac{t+2}{\sqrt{5}} + \sqrt{\left(\frac{t+2}{\sqrt{5}}\right)^2+1}\right) + C_2^0 =$$

$$= 2\ln\left(t+2+\sqrt{t^2+4t-1}\right) + C_2$$

$F_3(t) = -\int \frac{1}{t\sqrt{t \quad + 4t - 1}}dt$ položíme $z = \frac{1}{t}$ a volíme substituci:

$$t = \psi(z) = \frac{1}{z} \qquad \psi^{-1}(t) = \frac{1}{t} \qquad \psi'(z) = -\frac{1}{z^2}$$

$$J = (-2+2\sqrt{5}, \infty) \xrightarrow{\psi} J^0 = \left(0, \frac{1}{-2+2\sqrt{5}}\right)$$

$$G_3(z) = -\int \frac{z}{\sqrt{\frac{1}{z^2}+\frac{4}{z}-1}}\left(-\frac{1}{z^2}\right)dz = \int \frac{1}{\sqrt{1+4z-z^2}}\,dz = \frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{1-\left(\frac{z-2}{\sqrt{5}}\right)^2}}\,dz =$$

$$= \arcsin\frac{z-2}{\sqrt{5}} + C_3$$

$$G_3(\psi^{-1}(t)) = \arcsin\frac{1-2t}{t\sqrt{5}} + C_3 = F_3(t)$$

$$F(t) = F_1(t) + F_2(t) + F_3(t) = \frac{1}{2}\sqrt{t^2+4t-1} + 2\ln\left(t+2+\sqrt{t^2+4t-1}\right) + \arcsin\frac{1-2t}{t\sqrt{5}} + C$$

$$F(\varphi(x)) = \frac{1}{2}\sqrt{e^{2x}+4e^x-1} + 2\ln\left(e^x+2+\sqrt{e^{2x}+4e^x-1}\right) + \arcsin\frac{1-2e^x}{e^x\sqrt{5}} + C$$

3.1.133 $\int \frac{1}{\sqrt{1+e^x}+\sqrt{1-e^x}}\,dx \qquad D_f = (-\infty, 0\rangle$

upravíme:

$$\frac{1}{\sqrt{1+e^x}+\sqrt{1-e^x}} = \frac{\sqrt{1+e^x}-\sqrt{1-e^x}}{2e^x} = \frac{\sqrt{1+e^x}}{2e^x} - \frac{\sqrt{1-e^x}}{2e^x} = \frac{1+e^x}{e^{2x}}\frac{e^x}{2\sqrt{1+e^x}} + \frac{1-e^x}{e^{2x}}\frac{-e^x}{2\sqrt{1-e^x}}$$

$H_1(x) = \int \frac{1+e^x}{e^{2x}}\frac{e^x}{2\sqrt{1+e^x}}\,dx$, substituce: $t = \varphi(x) = \sqrt{1+e^x} \qquad I = (-\infty, 0) \xrightarrow{\varphi} J = (1,2)$

$$\varphi'(x) = \frac{e^x}{2\sqrt{1+e^x}} \qquad J = R - \{\pm 1\}$$

$$F_1(t) = \int \frac{t^2}{\left(t^2-1\right)^2}\,dt$$

nalezneme rozklad na parciální zlomky: $\dfrac{t^2}{\left(t^2-1\right)^2} = \dfrac{A}{t-1} + \dfrac{B}{(t-1)^2} + \dfrac{C}{t+1} + \dfrac{D}{(t+1)^2}$

t = -1 : $D = \dfrac{1}{4}$ $\quad t^3$: $A + C = 0$

t = 1 : $B = \dfrac{1}{4}$ $\quad t^0$: $A - C = \dfrac{1}{2}$ ……. $\Rightarrow A = \dfrac{1}{4}$, $C = -\dfrac{1}{4}$

$$F_1(t) = \frac{1}{4}\int\left(\frac{1}{t-1} + \frac{1}{(t-1)^2} - \frac{1}{t+1} + \frac{1}{(t+1)^2}\right)dt = \frac{1}{4}\ln\left(\frac{t-1}{t+1}\right) - \frac{1}{2}\frac{t}{t^2-1} + C_1$$

$$H_1(x) = \frac{1}{4}\ln\left(\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right) - \frac{1}{2}\frac{\sqrt{1+e^x}}{e^x} + C_1$$

$$H_2(x) = \int \frac{1-e^x}{e^{2x}}\frac{-e^x}{2\sqrt{1-e^x}}\,dx$$

zavedeme substituci: $t = \varphi(x) = \sqrt{1-e^x}$ $\qquad I = (-\infty, 0) \xrightarrow{\varphi} J = (0,1)$

$$\varphi'(x) = \frac{-e^x}{2\sqrt{1-e^x}} \qquad J = R - \{\pm 1\}$$

$$F_2(x) = \int \frac{t^2}{\left(1-t^2\right)^2}\,dt = \int \frac{t^2}{\left(t^2-1\right)^2}\,dt = \frac{1}{4}\ln\left(\frac{1-t}{1+t}\right) - \frac{1}{2}\frac{t}{t^2-1} + C_2$$

$$H_2(x) = \frac{1}{4}\ln\left(\frac{1-\sqrt{1-e^x}}{1+\sqrt{1-e^x}}\right) + \frac{1}{2}\frac{\sqrt{1-e^x}}{e^x} + C_2$$

$$H(x) = H_1(x) + H_2(x) = \frac{1}{2}\frac{\sqrt{1-e^x}-\sqrt{1+e^x}}{e^x} + \frac{1}{4}\ln\frac{\left(\sqrt{1+e^x}-1\right)\left(1-\sqrt{1-e^x}\right)}{\left(\sqrt{1+e^x}+1\right)\left(1+\sqrt{1-e^x}\right)} + C$$