3.3 Užití určitého integrálů v geometrii a fyzice

3.3.1 Vypočítejte obsah elipsy $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

a) při vyjádření explicitním
b) při vyjádření parametrickém

Řešení:

a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \rightarrow \quad \frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2}$

$$\Rightarrow \quad y^2=\frac{b^2}{a^2}\left(a^2-x^2\right)$$

$$\Rightarrow \quad y=\sqrt{\frac{b^2}{a^2}\left(a^2-x^2\right)}=\frac{b}{a}\sqrt{\left(a^2-x^2\right)}$$

$$P=2\cdot\int_{-a}^{a}\frac{b}{a}\sqrt{\left(a^2-x^2\right)}dx=\frac{2b}{a}\int_{-a}^{a}\sqrt{\left(a^2-x^2\right)}dx=\left|\begin{array}{c} x=a\sin t \\ dx=a\cos t\ dt \\ a\rightarrow\frac{\pi}{2} \\ -a\rightarrow-\frac{\pi}{2}\end{array}\right|=$$

$$=\frac{2b}{a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\left(a^2-a^2\sin^2 t\right)}\cdot a\cos t\ dt=\frac{2b}{a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}a\sqrt{\underbrace{\left(1-\sin^2 t\right)}_{\cos^2 t}}\cdot a\cos t\ dt=$$

$$=\frac{2b}{a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}a^2\cos^2 t\,dt=2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}dt=2ab\left[\tfrac{1}{2}t+\tfrac{1}{4}\sin 2t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=2ab\left[\left(\frac{\pi}{4}+\frac{\pi}{4}\right)\right]=\underline{ab\pi}$$

b) $x=a\sin t \quad \rightarrow \quad dx=a\cos t\ dt$

$y=b\cos t$

$$P=2\int_{-a}^{a}ydx=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}b\cos t\cdot a\cos t\ dt=2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 tdt=2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}dt=$$

$$=2ab\left[\frac{1}{2}t+\frac{1}{4}\sin 2t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\underline{ab\pi}$$

3.3.2 Vypočítejte délku oblouku paraboly $y=\frac{x^2}{2p}$ od jejího vrcholu po bod $\left[\sqrt{2p},1\right]$.

Řešení:

$$l=\int_{a}^{b}\sqrt{1+y'^2(x)}\,dx$$

$$y=\frac{x^2}{2p} \quad \rightarrow \quad y'=\frac{x}{p} \quad \rightarrow \quad y'^2=\frac{x^2}{p^2}$$

$$l=\int_a^b \sqrt{1+y'^2(x)}\,dx=\int_0^{\sqrt{2p}}\sqrt{1+\frac{x^2}{p^2}}dx=\left|\begin{array}{l}\frac{x}{p}=\sinh t\\ \frac{dx}{p}=\cosh t dt\end{array}\right|=\int\sqrt{1+\sinh^2 t}\cdot p\cdot\cosh t\,dt=$$

$$=p\int\cosh^2 t\,dt=p\cdot\left(\frac{1}{4}\sinh 2t+\frac{t}{2}\right)=p\cdot\left(\frac{1}{4}\sinh\left(2\operatorname{arg\,sinh}\tfrac{x}{p}\right)+\frac{\operatorname{arg\,sinh}\tfrac{x}{p}}{2}\right)_0^{\sqrt{2p}}=$$

$$=\frac{p}{4}\left(\sinh\left(2\operatorname{arg\,sinh}\tfrac{\sqrt{2p}}{p}\right)+2\operatorname{arg\,sinh}\tfrac{\sqrt{2p}}{p}\right)=\underline{\frac{p}{4}\left(\sinh\left(2\operatorname{arg\,sinh}\sqrt{\tfrac{2}{p}}\right)+2\operatorname{arg\,sinh}\sqrt{\tfrac{2}{p}}\right)}$$

3.3.3 Vypočítejte obsah a délku kardioidy $\rho=a\,(1+\cos\varphi),\ \varphi\in\langle 0\,,\,2\pi\rangle$.

Řešení:

$x=a(1+\cos\varphi)\cos\varphi$ $\qquad$ $x'=-a(\sin\varphi+2\sin\varphi\cos\varphi)$

$y=a(1+\cos\varphi)\sin\varphi$ $\qquad$ $y'=a(\cos\varphi-\sin^2\varphi+\cos^2\varphi)$

3.3.4 Vypočtěte obsah obrazce ohraničeného křivkami:

a) parabolou $y=6x-x^2$ a osou x.

$$P=\int_0^6(6x-x^2)dx=\left[\frac{6}{2}x^2-\frac{1}{3}x^3\right]_0^3=\underline{36}$$

b) parabolou $y^2=2px$ a přímkou $x-2y-1=0,\ p\rangle\,0$.

$$P=\int_0^1\left(\sqrt{2px}-\frac{x-1}{2}\right)dx=\sqrt{2p}\int_0^1 x^{\frac{1}{2}}dx-\frac{1}{2}\int_0^1(x-1)dx=\sqrt{2p}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1-\frac{1}{2}\left[\frac{x^2}{2}-x\right]_0^1=$$

$$=\underline{\sqrt{2p}\,\frac{2}{3}+\frac{1}{4}}$$