4.1 Řešení základních typů diferenciálních rovnic 1.řádu

4.1.1 Najděte obecné řešení diferenciální rovnice:

a) $tyy' = 1 - t^2$ b) $y' = 10^{t+y}$

Řešení:

a) $tyy' = 1 - t^2$

$ty\dfrac{dy}{dt} = 1 - t^2$

$tydy = \left(1 - t^2\right)dt$

$ydy = \left(\dfrac{1 - t^2}{t}\right)dt$

$\int ydy = \int\left(\dfrac{1 - t^2}{t}\right)dt$

$\dfrac{y^2}{2} = \ln t - \dfrac{t^2}{2} + \ln C$

$y^2 = 2\ln|ct| + t^2 \quad C \in R^+$

$\underline{y^2 + t^2 = \ln\left|ct^2\right|}$

b) $y' = 10^{t+y}$

$\dfrac{dy}{dt} = 10^t \cdot 10^y$

$\dfrac{1}{10^y}dy = 10^t\,dt$

$\int\dfrac{1}{10^y}dy = \int 10^t\,dt$

$-\dfrac{10^{-y}}{\ln 10} = \dfrac{10^t}{\ln 10} + C$

$\underline{10^{-y} + 10^t = C}$

4.1.2 Najděte obecné řešení homogenní diferenciální rovnice:

a) $(t + y)dt - (t - y)dy = 0$

b) $\left(t^2 + ty + y^2\right)dt = t^2 dy$

c) $y' = \dfrac{y^2}{t^2} - 2$

Řešení:

a) $(t + y)dt - (t - y)dy = 0$

$\dfrac{dy}{dt} = \dfrac{t + y}{t - y}$

$y' = \dfrac{dy}{dt} = \dfrac{1 + \frac{y}{t}}{1 - \frac{y}{t}}$ $\qquad y = \frac{y}{t}$ $\qquad \int\dfrac{1 - z}{1 + z}dz = \int\dfrac{1}{4}dt$

$$z't + z = \frac{1+z}{1-z}$$

$$z't = \frac{1+z}{1-z} - z$$

$$z't = \frac{1+z^2}{1-z}$$

$$y = z \cdot t$$
$$y' = z + z't$$

$$arctg\ z - \tfrac{1}{2}\ln\left(1+z^2\right) = \ln|t| + C$$

$$\underline{arctg\frac{y}{t} = \ln\sqrt{t^2+y^2} + C}$$

b) $\left(t^2 + ty + y^2\right)dt = t^2 dy$

$$\frac{dy}{dt} = \frac{t^2 + ty + y^2}{t^2}$$

$$z = \tfrac{y}{t} \quad \rightarrow y = z \cdot t, \quad y' = z + z't$$

$$y' = 1 + \frac{y}{t} + \frac{y^2}{t}$$

$$z't + z = 1 + z + z^2$$

$$z't = 1 + z^2$$

$$\frac{z'}{1+z^2} = \frac{1}{t}$$

$$\int \frac{z'}{1+z^2} = \int \frac{1}{t}$$

$$arctg\ z = \ln|t| + \ln|C|$$

$$arctg\frac{y}{t} = \ln t \cdot C$$

$$\underline{y = t \cdot tg(\ln Ct)}$$

c) $y' = \dfrac{y^2}{t^2} - 2$

$$u't + u = u^2 - 2$$

$$u't = u^2 - u - 2$$

$$\frac{du}{dt} = \frac{u^2 - u - 2}{t}$$

$$\frac{du}{(u-2)(u+1)} = \frac{dt}{t}$$

$$u = \tfrac{y}{t} \rightarrow y = u \cdot t$$

$$\frac{1}{3}\int\frac{du}{u-2} - \frac{1}{3}\int\frac{du}{u+1} = \int\frac{dt}{t}$$

$$\tfrac{1}{3}\ln(u-2) - \tfrac{1}{3}\ln(u+1) = \ln|t| + \ln|c|$$

$$\ln\left|\frac{u-2}{u+1}\right| = \ln Ct^3$$

$$\frac{u-2}{u+1} = Ct^3$$

$$\frac{\frac{y}{t} - 2}{\frac{y}{t} + 1} = Ct^3 \quad \rightarrow \quad \underline{y - 2t = Ct^3(y+t)}$$

4.1.3 Najděte obecné řešení diferenciální rovnice:

a) $y''' = y''^3$

b) $ay''' = y''$

Řešení:

a) $y''' = y''^3$

$\rightarrow \quad z = y'',\ z' = y'''$

$z' = z^3$

$\frac{dz}{z^3} = dt$

$\int \frac{dz}{z^3} = \int dt$

$-\frac{1}{2z^2} = t + C \qquad C \in R$

$z^2 = -2t + C_1 \qquad C_1 \in R$

$z = (C_1 - 2t)^{-\frac{1}{2}}$

$y'' = (C_1 - 2t)^{-\frac{1}{2}} = \sqrt{(C_1 - 2t)^{-1}}$

$y' = \int (C_1 - 2t)^{-\frac{1}{2}} dt$

$\underline{y = (C_1 - 2t)^{\frac{3}{2}} + C_2 t + C_3}$

b) $ay''' = y''$

$\rightarrow \quad z = y'',\ z' = y'''$

$az' = z$

$\int \frac{a}{z} dz = \int dt$

$a \cdot \ln|z| = t + \ln|C_1|$

$\ln|z| = \frac{t + C}{a}$

$y = \pm e^{\frac{t+C}{a}} \qquad C_1 = \pm e^{\frac{C}{a}} \in R - \{0\}$

$y' = \pm e^{\frac{t+C}{a}} + C_2 t + C_3$

$y = \pm e^{\frac{C}{a}} \cdot e^{\frac{t}{a}} + C_2 t + C_3 \qquad C_2, C_3 \in R$

$\underline{y = C_1 \cdot e^{\frac{t}{a}} + C_2 t + C_3}$

4.1.4 Najděte obecné řešení diferenciální rovnice

a) $(2t + 2y - 1) + y'(t + y - 2) = 0$

b) $(t - y + 2)dt + (y - t + 3)dy = 0$

c) $(t - 2y + 5)dt + (2t - y + 4)dy = 0$

Řešení:

a) $(2t + 2y - 1) + y'(t + y - 2) = 0$

$$y' = -\frac{2t + 2y - 1}{t + y - 2}$$

$$y' = -\frac{t+y-2+t+y+1}{t+y-2} \qquad t+y-2=z$$

$$y = z-t+2$$

$$y' = -1-\frac{t+y+1}{t+y-2} \qquad y' = z'-1$$

$$z'-1 = -1-\frac{t+y+1}{t+y-2}$$

$$z'-1 = -1-\frac{z+3}{z} \qquad -\int\frac{z}{z+3} = \int dt$$

$$z' = -\frac{z+3}{z} \qquad -z+3\ln|z+3| = t+C_1$$

$$\frac{dz}{dt} = -\frac{z+3}{z} \qquad 3\ln|z+3| = t+z+C_1$$

$$\ln|z+3| = \frac{t}{3}+\frac{z}{3}+\frac{C_1}{3}$$

$$\ln|t+y+1| = \ln e^{\frac{2t+y}{3}} + \ln e^{c_1-\frac{2}{3}}$$

$$\underline{t+y+1 = C\cdot e^{\frac{2t+y}{3}}}$$

b) $(t-y+2)dt+(y-t+3)dy=0$ $\qquad y-t+3=z$

$$(t-y+2)dt = -(y-t+3)dy \qquad y = z+t-3$$

$$-\frac{dy}{dt} = \frac{t-y+2}{y-t+3} \qquad y' = z'+1$$

$$y' = -\frac{t-y+2}{y-t+3}$$

$$y' = 1-\frac{5}{y-t+3}$$

$$z'+1 = 1-\frac{5}{z} \qquad \int zdz = -5\int dt$$

$$z' = -\frac{5}{z} \qquad \frac{1}{2}z^2 = -5t+C_1$$

$$dz\cdot z = -5\cdot dt \qquad z^2 = -10t+2C_1 \qquad 2C_1 = C$$

$$(y-t+3)^2 = -10t+C$$

$$y^2-yt+3y-ty+t^2-3t+3y-3t+9+10t = C$$

$$\underline{(y-t)^2+6y+4t = C}$$

c) $(t-2y+5)dt = -(2t-y+4)dy$

$$y' = -\frac{t-2y+5}{2t-y+4} \qquad t = u+A \qquad 2A-B=-4$$

$$y = v+B \qquad \underline{A-2B=-5}$$

$$\frac{dv}{du} = -\frac{u-2v}{2u-v}$$

$$A = -1$$

$$B = 2$$

$$\frac{dv}{du} = -\frac{1-2\frac{v}{u}}{2-\frac{v}{u}} \qquad \frac{v}{u} = z$$

$$z' \cdot u + z = -\frac{1-2z}{2-z} \qquad v = z \cdot u$$

$$z' \cdot u = -\frac{1+2z-2z+z^2}{2-z} \qquad v' = z' \cdot u + z$$

$$\int \frac{du}{u} = \int \frac{2-z}{z^2-1} dz$$

$$z' \cdot u = \frac{-1+z^2}{2-z} \qquad \ln u - \ln C = -\frac{3}{2}\ln|z+1| + \frac{1}{2}\ln|z-1|$$

$$\frac{du}{u} = \frac{2-z}{z^2-1} dz \qquad \ln u + \ln C = -\frac{2}{3}\ln\left|\frac{v}{u}+1\right| + \frac{1}{2}\ln\left|\frac{v}{u}-1\right|$$

$$\ln|t+1| + \ln C = -\frac{2}{3}\ln\left|\frac{y-2}{t+1}+1\right| + \frac{1}{2}\ln\left|\frac{y-2}{t+1}-1\right|$$

$$\sqrt{\frac{\frac{y-t-3}{t+1}}{\left(\frac{y+t-1}{t+1}\right)^3}} = (t+1)C_1$$

$$\frac{y-t-3}{y+t-1} = (y+t-1)^2 C_1$$

$$\underline{(t+y-1)^3 = C \cdot (t-y+3)}$$

4.1.5 Najděte obecné řešení diferenciální rovnice:

$$y' + 2y = t^2 + 2t$$

__Řešení__:

$$y' + 2y = t^2 + 2t \quad \Rightarrow \quad y' = t^2 - 2t - 2y$$

$$y' = -2y \qquad -2e^{-2t} \cdot K + e^{-2t} \cdot K' + 2e^{-2t} \cdot K = t^2 + 2t$$

$$\frac{dy}{dt} \cdot \frac{1}{y} = -2 \qquad K' \cdot e^{-2t} = t^2 + 2t$$

$$\int \frac{dy}{y} = -2\int dt \qquad \frac{dK}{dt} = (t^2 + 2t)e^{+2t}$$

$$\ln y = -2t + \ln C \qquad K = \int t^2 e^{2t} dt + \int 2te^{2t} dt$$

$$y = e^{-2t} \cdot K \qquad K = \frac{1}{4}e^{2t}(2t^2 + 2t - 1) + C$$

$$y' = -2e^{-2t} \cdot K \cdot e^{-2t} \qquad y = e^{-2t\cdot} \cdot \frac{1}{4}e^{2t}(2t^2 + 2t - 1)$$

$$\underline{y = Ce^{-2t} + \frac{1}{4}\left(2t^2 + 2t - 1\right)}$$

$$\int t^2 e^{2t} dt = \begin{vmatrix} u = t^2 & v' = e^{2t} \\ u' = 2t & v = \frac{1}{2}e^{2t} \end{vmatrix} = t^2 \frac{1}{2}e^{2t} - \int 2t \cdot \frac{1}{2}e^{2t} dt =$$

$$= t^2 \frac{1}{2}e^{2t} - \int t \cdot e^{2t} dt = \begin{vmatrix} u = t & v' = e^{2t} \\ u' = 1 & v = \frac{1}{2}e^{2t} \end{vmatrix} =$$

$$= \frac{1}{2}t^2 e^{2t} - t \cdot \frac{1}{2}e^{2t} - \frac{1}{2}\int e^{2t} dt = \frac{1}{2}te^{2t}(t-1) + \frac{1}{4}e^{2t} + C$$

$$\int 2te^{2t} dt = \begin{vmatrix} u = t & v' = e^{2t} \\ u' = 1 & v = \frac{1}{2}e^{2t} \end{vmatrix} = 2\left(t \cdot \frac{1}{2}e^{2t} - \frac{1}{2}\int e^{2t} dt\right) = t \cdot e^{2t} - \frac{1}{2}e^{2t} + C$$

$$\frac{1}{2}te^{2t}(t-1) + \frac{1}{4}e^{2t} + t \cdot e^{2t} - \frac{1}{2}e^{2t} + C =$$

$$= \frac{1}{2}t^2 e^{2t} - \frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} + t \cdot e^{2t} - \frac{1}{2}e^{2t} + C = \frac{1}{4}e^{2t}\left(2t^2 + 2t - 1\right)$$

4.1.6 Najděte obecné řešení Bernoulliovy diferenciální rovnice:

a) $$y' + \frac{y}{t+1} = -\frac{1}{2}(t+1)^3 y^3$$

b) $$y' + \frac{2y}{t} = \frac{2\sqrt{y}}{\cos^2 t}$$

__Řešení__:

a) $$y' - \frac{y}{t+1} = -\frac{1}{2}(t+1)^3 y^3$$

$$y' = -\frac{1}{2}(t+1)^3 y^3 - \frac{y}{t+1} \quad \Big| : y^3$$

$$\frac{y'}{y^3} = -\frac{1}{2}(t+1)^3 - \frac{1}{y^2(t+1)}$$

$$\frac{1}{y^2} = z \Rightarrow z' = -\frac{2}{y^3} \cdot y' - \frac{z'}{z} = \frac{y'}{y^3}$$

$$\frac{-z'}{2} = -\frac{1}{2}(t+1)^3 - \frac{z}{(t+1)}$$

$$C = C(t)$$

$$z' = (t+1)^3 - \frac{2z}{t+1}$$

$$z' = C'(t) \cdot (t+1)^2 + 2(t+1)C(t)$$

$$z' = (t+1)^3 + 2(t+1) \cdot C(t)$$

$$(t+1)^3 + 2(t+1) \cdot C(t) = C'(t)(t+1)^2 + 2(t+1) \cdot C(t)$$

$$C'(t) = t+1$$

$$C(t) = \frac{1}{2}(t+1)^2 + C$$

$$z = (t+1)^2\left(\frac{3}{2}(t+1) + C\right)$$

$$\underline{\frac{1}{y^2} = \frac{1}{2}(t+1)^4 + C(t+1)^2}$$

b) $y' + \frac{2y}{t} = \frac{2\sqrt{y}}{\cos^2 t} \quad \Big|: \sqrt{y}$

$\frac{y'}{\sqrt{y}} + \frac{2\sqrt{y}}{t} = \frac{2}{\cos^2 t}$ $\quad\quad \left[z = \sqrt{y} \quad z' = \frac{1}{2}\frac{y'}{\sqrt{y}} \right]$

$2z' + \frac{2z}{t} = \frac{2}{\cos^2 t}$

$2z' + \frac{2z}{t} = 0$

$z' + \frac{z}{t} = 0$

$z = \frac{K}{t} \quad\quad K \in R$

$z' = \frac{K't - K}{t^2}$

$2\frac{K't - K}{t^2} + 2 \cdot \frac{K}{t} = \frac{2}{\cos^2 t}$

$\frac{2K'}{t} - \frac{2K}{t^2} + \frac{2K}{t^2} = \frac{2}{\cos^2 t}$

$\frac{K'}{t} = \frac{1}{\cos^2 t}$

$K' = \frac{t}{\cos^2 t}$

$K = t \cdot tg\, t + \ln|\cos t| + C \quad\quad C \in R$

$z = tg\, t + \frac{\ln|\cos t| + C}{t}$

$\underline{y = \left(tg\ t + \frac{\ln \cos t + C}{t} \right)^2}$

$\underline{y = 0}$

Řešení nehomogenní diferenciální rovnice $y' + \frac{2y}{t} = \frac{2\sqrt{y}}{\cos^2 t}$ je $y = \left(tg\, t + \frac{\ln \cos t + C}{t} \right)^2$ + singulární řešení y = 0.

4.1.7 Diferenciální rovnici $(y - t)y' = \frac{y^2}{t}$ řešte

a) jako homogenní, určete také partikulární řešení dané počáteční podmínkou $y(1) = -1$

b) jako Bernoulliovu přechodem k inverzní funkci $t(y)$

__Řešení__:

a) $y' = \frac{y^2}{(y - t)t}$

$y' = \frac{y}{ty'} - \frac{y^2}{t^2}$

$z't + z = z(z't + z) - z^2$

$z't + z = zz't + z^2 - z^2$

$z'(t + zt) = -z$

$z' = -\frac{z}{(t + zt)}$

$\frac{y}{t} = zy' = z't + z$

$\int \frac{1+z}{z} dz = -\int \frac{dt}{t}$

$\ln|z| + z = -\ln|t| + C_1$

$-z - \ln|z| = \ln|t| + C_1$

$e^z = z \cdot t \cdot C_2 \quad \Rightarrow \quad \underline{e^{\frac{y}{t}} = y \cdot C}$

$$z' = -\frac{z}{t(1+z)}$$

b) $$y' = \frac{y^2}{(y-t)t}$$

$$t' = \frac{(y-t)t}{y^2}$$

$$t' = \frac{t}{y} - \frac{t^2}{y^2} \quad \Big| : t^2$$

$$\frac{t'}{t^2} = \frac{1}{ty} - \frac{1}{y^2}$$

$$\frac{1}{t} = z \Rightarrow z' = -\frac{1}{t^2} \cdot t'$$

$$z' = \frac{z}{y} - \frac{1}{y}$$

$$\ln e^{\frac{y}{t}} = \ln y \cdot C$$

$$\underline{Cy = e^{\frac{y}{t}}}$$

$$z' = -\frac{z}{y}$$

$$\int \frac{dz}{z} = -\int \frac{dy}{y}$$

$$\ln|z| = -\ln|y| + \ln C$$

$$z = \frac{C}{y}$$

$$C = C(y) z' = C'(y) \cdot \frac{1}{y} + C(y) \cdot \left(-\frac{1}{y^2} \right)$$

$$z' = \frac{1}{y^2} - C(y) \cdot \frac{1}{y^2}$$

$$C'(y) = \frac{1}{y} \quad ; \quad C(y) = \ln|y| + C$$

$$z = \frac{\ln y + C}{y}$$

$$\frac{1}{t} = \frac{\ln y + C}{y}$$